THE PRINCIPLES AND PRACTICE OF ARITHMETIC,

COMPRISING

THE NATURE AND USE OF LOGARITHMS,

WITH

THE COMPUTATIONS EMPLOYED BY ARTIFICERS, GAGERS, AND LAND-SURVEYORS.

DESIGNED FOR THE USE OF STUDENTS.

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ADVERTISEMENT.

In the present Treatise it has been the Author's endeavour to combine what is necessary of the *Philosophy* of the *Science of Arithmetic* with the *Practice* of the *Art of Numbers*: and it is here considered sufficient to place before the student an outline of the plan which has been adopted in the arrangement, with a short account of the more important divisions.

The *first* Chapter commences with the elementary Definitions; it then proceeds to the explanation of *Notation* and *Numeration*, which are both exemplified in a great variety of instances; and concludes with the consideration of the *Fundamental Operations* of the Science as applied to *pure* or *abstract* numerical magnitudes.

In the *second* Chapter, the Fundamental Operations have been extended to *mixed* or *concrete* numerical magnitudes, consisting of various denominations; and some important remarks are introduced.

The *third* Chapter treats of the First Principles of *The Rule of Three*, sometimes called *The Golden Rule;* and comprises a collection of examples intended to illustrate its different views.

The *fourth* Chapter contains *The Doctrine of Fractions,* usually termed *Vulgar Fractions;* and concludes with many of their applications to practical purposes.

The *fifth* Chapter developes *The Theory of Decimals,* commonly called *Decimal Fractions;* and points out most of the uses to which Decimals are peculiarly adapted.

In the *sixth* Chapter are discussed the Doctrines of *Ratio* and *Proportion,* from the *Principles* of which are deduced all the Rules of consequence in the affairs of *Commerce;* and it concludes with the solution of a few miscellaneous questions, explaining some *technical* terms.

The *seventh* Chapter contains the Practice of *Involution* and *Evolution,* with *The Arithmetic of Surds* or *Irrational Quantities.*

The object of the *eighth* Chapter is *The Nature and Properties of Logarithms,* derived from the simplest principles; and the *practical* advantages afforded by *Logarithmic Tables* are pointed out in appropriate examples.

The *ninth* Chapter is *The Application of Arithmetic to Geometry:* and the Calculations of *Artificers, Gagers* and *Land-Surveyors,* are explained and exemplified in it.

In this Chapter will also be found an account of the *Imperial Weights* and *Measures,* and their origin and relation to each other; as well as of the *Calendar* adopted in the time of *Julius Cæsar,* and its subsequent improvement in the time of *Pope*

Gregory the *Thirteenth*, with the requisite Calculations worked out.

The rest is an Appendix, in which the *Fundamental Rules* have been derived from *Elementary Principles*, upon the extension of which the present system of Arithmetic is established.

Throughout the work, it has been attempted to trace the source of every Rule which is given, and to investigate the reasons upon which it is founded: and by means of *particular* examples comprising nothing but what is *common* to every other example of the *same kind*, to attain in Arithmetic the kind of evidence which is relied upon in Geometry, or in any other demonstrative Science.

Single and *Double Position* are omitted, as most of the examples usually given to illustrate these rules, may be solved by the principles here explained, not to mention that they are merely *Algebraical Formulæ* enunciated at length.

No notice has been taken of *Arithmetical* and *Geometrical Progression*, of *Permutations* and *Combinations*, and of *Annuities* and *Reversions*, because they depend upon *Formulæ* expressed by general symbols, which the student would find a difficulty in making use of, without at least a knowledge of the *Notation* and *Fundamental Operations* of *Algebra*; in addition to which, they seldom occur to any one who is not engaged in *Scientific* Speculations, or in *Professional* Calculations.

It has been objected that the *Examples for Practice* given in this work, are too numerous for a rapid advancement in the subject; but the student

will recollect that he has no occasion to trouble himself with the *rest,* when a *few* of them have rendered him perfect in the Application of the Rules ; although it must be observed, that a *Facility* in Arithmetical Calculations is of all things the most indispensable, in the formation both of the future *Analyst*, and of the *Man of Business.*

CAMBRIDGE, 1849.

TABLES

OF

MONEY, WEIGHTS AND MEASURES,

WITH SOME OBSERVATIONS RESPECTING THEM.

I. Table of Money or Value.

A Farthing is written or marked $\frac{1}{4}d.$

2 Farthings are	1	Halfpenny	$\frac{1}{2}d.$
4 Farthings	1	Penny	1*d.*
12 Pence	1	Shilling	1*s.*
20 Shillings	1	Pound	£1.

Money as expressed by means of these denominations is called *Sterling* money, in order to distinguish it from *Stocks, Shares,* &c.

The *Standard* gold coin of this Kingdom is made of a metal consisting of 22 parts of *pure gold,* and 2 parts of *copper.* The Pound sterling is represented by a gold coin called a *Sovereign,* and from a pound troy of standard gold are coined $46\frac{29}{40}$ sovereigns, so that the weight of each is 5dwts. $3\frac{171}{623}$grs., or 123.274grs.; and the value of gold of the *Mint-Fineness,* called 22 *Carat* gold, is £3. 17*s.* $10\frac{1}{2}d.$ per ounce.

The *Standard* silver coin consists of 37 parts of *pure silver,* and 3 parts of *copper:* and a pound troy of this metal furnishes 66 *shillings,* so that the weight of a shilling is 3dwts. $15\frac{3}{11}$grs., and the *Mint-Price* of standard silver is 5*s.* 6*d.* per ounce. The silver coinage is not a *legal tender* for more than 40s., the gold coinage being the *general* standard of value.

In the copper Coinage, 24 *pence* are made from an avoirdupois pound of copper, so that a penny should weigh $10\frac{2}{3}$drs. avoirdupois, or $291\frac{2}{3}$grs. troy: and this coin is not a legal tender for more than 12*d.*

A *Farthing,* or 1*f.,* is the lowest denomination above mentioned: but farthings and all other subdivisions of a penny are in practice, denoted by *Fractions* of a Penny.

The Coins now *current* are the following, constituting what is called *The Circulating Medium.*

Copper.

A Half-farthing . . .	$\frac{1}{2}f. = \frac{1}{8}d.$
A Farthing	$1f. = \frac{1}{4}d.$
A Half-penny	$2f. = \frac{1}{2}d.$
A Penny	$4f. = 1d.$

Silver.

A Fourpence	= 0 . 4.
A Sixpence	= 0 . 6.
A Shilling	= 1 . 0.
A Half-crown	= 2 . 6.
A Crown	= 5 . 0.

Gold.

	£.	s.
A Half-sovereign . . =	0 .	10.
A Sovereign . . . . . =	1 .	0.
A Double-sovereign. =	2 .	0.

The following coins, no longer in circulation, are frequently mentioned, and their values are subjoined.

	£.	s.	d.		£.	s.	d.
A Groat . . . =	0 .	0 .	4.	An Angel . . =	0 .	10 .	0.
A Tester . . . =	0 .	0 .	6.	A Mark . . . =	0 .	13 .	4.
A Guinea . . =	1 .	1 .	0.	A Carolus . . =	1 .	3 .	0.
A Half-guinea =	0 .	10 .	6.	A Jacobus . . =	1 .	5 .	0.
A Noble . . . =	0 .	6 .	8.	A Moidore . . =	1 .	7 .	0.

II. Table of Avoirdupois Weight.

A Dram is written 1 dr.

16 Drams are	1 Ounce	1	oz.
16 Ounces	1 Pound	1	lb.
14 Pounds	1 Stone	1	st.
2 Stone or 28 lbs	1 Quarter	1	qr.
4 Quarters or 112lbs	1 Hundredweight .	1	cwt.
20 Hundredweight	1 Ton	1	ton.

A Firkin of Butter is 4 stone or 56 lbs.; a Fodder of Lead is $19\frac{1}{2}$ cwt.: and several sorts of Silk are sometimes weighed by what is called a *great pound* of 24 ounces.

By this table are computed the weights of all substances which constitute the Necessaries of Life and the Objects of Commerce, with all Metals, except *Gold, Silver* and some *Precious Stones.* See Article (223).

Wool Weight.

7 Pounds are .	1 Clove . . .	1 cl.
2 Cloves . . : .	1 Stone . . .	1 st.
2 Stone	1 Tod	1 tod.
$6\frac{1}{2}$ Tods	1 Wey . . .	1 wey.
2 Weys	1 Sack . . .	1 sack.
12 Sacks	1 Last . . .	1 last.
240 Pounds . . .	1 Pack . . .	1 pack.

III. Table of Troy Weight.

A Grain is written 1 gr.

24 Grains are . . .	1 Pennyweight . . .	1 dwt.
20 Pennyweights .	1 Ounce	1 oz.
12 Ounces	1 Pound	1 lb.

This weight is applied to gold, silver, common jewels, liquors, &c.: and is used in Philosophical Experiments.

The weights of the more valuable precious stones are estimated by the *Carat* of $3\frac{1}{6}$ grains Troy, the carat being divided into 4 *grains*, each of which is therefore equal to $\frac{19}{24}$ of a grain Troy. See Article (224).

Apothecaries Weight.

20 Grains are . .	1 Scruple . . .	1 scr. or 1 ℈.
3 Scruples . . .	1 Dram	1 dr. or 1 ʒ.
8 Drams	1 Ounce . . .	1 oz. or 1 ℥.
12 Ounces	1 Pound	1 lb. or 1 ℔.

These subdivisions are employed by Apothecaries in making up Medical Prescriptions, in which the latter *Symbols* are used; and the pound, ounce and grain are the same as in Troy-weight. See Article (224).

IV. Table of Lineal Measure.

An Inch is written 1 in.

12	Inches are	1 Foot . . .	1 ft.
3	Feet	1 Yard . . .	1 yd.
$5\frac{1}{2}$	Yards	1 Pole . . .	1 po.
4	Poles or 22 yds	1 Chain . .	1 ch.
40	Poles or 220 yds	1 Furlong .	1 fur.
8	Furlongs or 1760 yds . .	1 Mile . . .	1 mi.

By this measure are estimated the lineal dimensions of all magnitudes, with the exception mentioned below. See Article (221).

Cloth Measure.

4 Nails are	1 Quarter	1 qr.
4 Quarters	1 Yard	1 yd.

This measure is used for all kinds of Cloth: and a Nail, being a *sixteenth* part of 1 yard or of 36 inches, is therefore equal to $2\frac{1}{4}$ inches. An *Ell* is 5 quarters in *England*, but the *Flemish* and *French* Ells are nearly equal to 3 and 6 *English* quarters respectively.

To these the following table may be annexed, exhibiting the magnitudes of certain measures frequently mentioned in books, or used on particular occasions.

A Line is	$\frac{1}{12}$ Inch.
A Barleycorn	$\frac{1}{3}$ Inch.
A Palm	3 Inches.
A Hand	4 Inches.
A Span	9 Inches.
A Cubit	18 Inches.
A Pace	5 Feet.
A Fathom	6 Feet.
A Rod or Perch	$5\frac{1}{2}$ Yards.
A League	3 Miles.
A Geographical Degree .	$69\frac{1}{9}$ Miles.

A Link, being a *hundredth* part of a Chain, is $7\frac{23}{25}$ inches; and a Geographical Mile is a *sixtieth* part of a Geographical Degree.

A *Barleycorn* or *Grain* of *Barley* is supposed to have been the original Element of Lineal Measure, in the same manner as a *Grain* of *Wheat* is said to have given rise to the name of the Element of Weight.

V. Table of Superficial Measure.

A Square Inch is written 1 sq. in. or 1 in.

144 Square Inches are	1 Square Foot	.	1 sq. ft. or 1 ft.
9 Square Feet . . .	1 Square Yard	.	1 sq. yd. or 1 yd.
$30\frac{1}{4}$ Square Yards . .	1 Square Pole	.	1 sq. po. or 1 po.

A *Square Rod* of $272\frac{1}{4}$ square feet is used in estimating Bricklayers' work: and a *Square* of Flooring, Roofing, &c., is 100 square feet. See Article (203), &c.

Land Measure.

40 Poles are	1 Rood	. . . .	1 ro.
4 Roods . .	1 Acre	. . . .	1 ac.

Also, 1210 sq. yds., or 25000 sq. links = 1 Rood:
4840 sq. yds., or 100000 sq. links = 1 Acre.

A *Yard* of land, and a *Hide* of land, terms often used in *former times*, are generally supposed to have been equivalent to 30 acres and 100 acres respectively, of the modern admeasurement. See Article (219).

VI. Table of Solid Measure.

A Cubic Inch is written 1 cu. in. or 1 in.

1728 Cubic Inches are	1 Cubic Foot . .	1 ft.
27 Cubic Feet . . .	1 Cubic Yard .	1 yd.

A Load of *rough* timber is 40 cubic feet; a Load of *squared* timber is 50 cubic feet; and a Ton of *Shipping* is 42 cubic feet. See Article (208), &c.

VII. Table of Measure of Capacity.

A Gill is written 1 gil.

4 Gills are	1 Pint . . .	1 pt.
2 Pints . .	1 Quart . .	1 qt.
4 Quarts . .	1 Gallon . .	1 gal.

By this measure liquids, corn, seeds, lime, &c., are estimated according to the multiples in the following Tables. See Article (222).

Ale and Beer Measure.

9 Gallons are	1 Firkin	1 fir.
2 Firkins or 18 gals.	1 Kilderkin	1 kil.
2 Kilderkins or 36 gals.	1 Barrel	1 bar.
$1\frac{1}{2}$ Barrels or 54 gals.	1 Hogshead	1 hhd.
2 Hogsheads	1 Butt	1 butt.
2 Butts	1 Tun	1 tun.

Wine and Spirit Measure.

10 Gallons are	1 Anker	1 ank.
18 Gallons	1 Runlet	1 run.
42 Gallons	1 Tierce	1 tier.
2 Tierces	1 Puncheon	1 pun.
63 Gallons	1 Hogshead	1 hhd.
2 Hogsheads	1 Pipe	1 pipe.
2 Pipes	1 Tun	1 tun.

Corn and Seed Measure.

2 Quarts are	1 Pottle	1 pot.
2 Pottles	1 Gallon	1 gal.
2 Gallons	1 Peck	1 pk.
4 Pecks	1 Bushel	1 bush.
2 Bushels	1 Strike	1 str.
2 Strikes or 4 Bushels	1 Coom	1 coom.
2 Cooms or 8 Bushels	1 Quarter	1 qr.
5 Quarters or 40 Bushels	1 Load	1 load
2 Loads or 10 Quarters	1 Last	1 last.

A Sack of Flour is a quantity which weighs 20 stone or 280 lbs., and it is generally about 5 Imperial Bushels. See Article (222).

Coal Measure.

4 Pecks are	1 Bushel.
3 Bushels	1 Sack.
36 Bushels	1 Chaldron.
21 Chaldrons	1 Score.

This table is of little use, as Coals are now sold by weight. See Article (222).

VIII. Table of Measure of Time.

A Second is written 1 sec. or 1″.

60 Seconds are . . .	1 Minute . . .	1 min. or 1′.
60 Minutes	1 Hour	1 hr.
24 Hours	1 Day	1 day.
7 Days	1 Week . . .	1 wk.

On some occasions 28 days, which is nearly a *Lunar Month*, are called a Month: and a common year consists of 12 *Calendar* Months: or of 12 *Average* Months of $30\frac{5}{12}$ days, nearly: or of 365 days, 6 hours: or of 52 weeks, 1 day, 6 hours: or of 13 lunar months, 1 day, 6 hours, nearly: the odd day and hours being omitted in *practice*; and the numbers of days in the *Calendar* Months are recollected by means of the following lines.

Thirty days have September,
April, June and November:
February twenty-eight alone:
And all the rest have thirty-one;
Except in Leap-year, and then is the time,
When February's days are twenty-nine.

For the History of the Calendar, see Article (227), &c.

IX. Table of Angular Measure.

A Second is written 1 sec. or 1″.

60 Seconds are	1 Minute	1 min, or 1′.
60 Minutes . .	1 Degree	1 deg. or 1°.
90 Degrees . .	1 Right Angle . .	1 rt. ang. or 90°.

There are also denominations below seconds, called *thirds, fourths*, &c., each being one *sixtieth* part of that which precedes it; but they are generally expressed *decimally* as parts of a Second. See Article (226).

X. Table of Number, &c.

12 Units are	1 Dozen.
12 Dozen	1 Gross.
20 Units	1 Score.
24 Sheets of *Paper* . . .	1 Quire.
20 Quires	1 Ream.
2 Reams	1 Bundle.
5 Bundles	1 Bale.

A *long* hundred is 120; a *great* gross is 144 dozen: but these, and many other denominations of a similar kind, are rapidly going out of use.

The Student will consult his own advantage and convenience by committing these ten tables to *memory*, omitting such of the observations as may depend upon principles beyond the extent of his progress in the subject.

THE

PRINCIPLES AND PRACTICE

OF

ARITHMETIC.

CHAPTER I.

DEFINITIONS, PRELIMINARY NOTIONS, NOTATION, NUMERATION, AND FUNDAMENTAL OPERATIONS.

ARTICLE I. DEFINITION I.

ARITHMETIC is the Science which treats of magnitudes, with reference to the consideration of *how many* or *how few*.

2. DEF. 2. An *Unit*, or, as it is generally called, *Unity*, is the representation of a thing considered in its *individual* capacity, without regard to the *parts* of which it may be made up, and it is the *Base* or *Element* of all arithmetical computations.

Thus, each of the terms, *a* man, *a* house, *a* pound, &c., denotes one individual of its kind, being the same as *one* man, *one* house, *one* pound, &c., respectively; and these are the bases or elements by means of which *several* men, *several* houses, *several* pounds, &c., may be computed.

3. DEF. 3. *Number* signifies a multitude or collection of *two* or *more* units, or denotes an *assemblage* of two or more *distinct* objects of the same kind.

Thus, *two* men, *three* houses, *four* pounds, &c., which are represented by the numbers, *two, three, four, &c.* denote more than one individual of the same kind, the single individuals being repeated *twice, thrice, four times, &c.*, respectively. Numbers thus viewed are termed *Whole Numbers* or *Integers*; and for the sake of uniformity, the Unit is considered the first or least integer.

4. Def. 4. Numbers used to express one or more individuals of *specified* kinds, are called *applicate* or *concrete* numbers; whereas *two*, *three*, *four*, *&c.*, by themselves, not particularizing the *kinds* of individuals, are termed *abstract* numbers.

NOTATION.

5. Def. 1. *Notation* is the method of expressing by certain symbols, or characters, any proposed number, or quantity arithmetically considered.

6. Def. 2. The *Symbol* or *Representative* of unit or unity, is 1; but instead of other numbers being expressed by assemblages or multitudes of units *placed* together, which would soon become embarrassing, other characters or symbols have been invented, by means of which every number however great, may be expressed; and instead of a different symbol being adopted for every different number, which would soon become equally inconvenient, *all* numbers are expressed by means of the following *ten* symbols, or as they are usually termed *Figures*, and sometimes *Digits*, which have their names respectively annexed:

1,	2,	3,	4,	5,	6,	7,	8,	9,	0:
one,	two,	three,	four,	five,	six,	seven,	eight,	nine,	zero:

the first *nine* of which are all defined by their names; and the *last* which is variously denominated *Nought*, *Cipher*, or *Zero*, when standing by itself has no signification, or at most, denotes the absence of number, and is to be regarded merely as an *auxiliary* digit, for the purposes hereafter to be explained.

7. Def. 3. Whenever a figure is placed on the *right* of the same or any other figure, it has, by *universal agreement*, the effect of increasing the value of the last-mentioned figure *tenfold*, at the same time that it retains its own value.

Thus, beginning with the auxiliary digit 0, we have the following numbers and their representations:

10,	11,	12,	13,	14,	&c.
ten,	eleven,	twelve,	thirteen,	fourteen,	&c.

20,	21,	22,	&c.
twenty	twenty-one,	twenty-two,	&c.

and it is obvious that by means of *two* figures, this kind of notation may be continued till we arrive at *ninety-nine,* whose symbol will be 99.

8. Def. 4. Beyond this number, the use of *two,* either the *same* or *different* figures, will not enable us to go, but a repetition of the contrivance in the last Article, will by means of *more* figures supply the defect.

Thus, supposing the effect of any figure's being placed on the right of symbols formed as above, to be to increase all their values *tenfold,* we shall have

100,	101,	102,	&c.
one hundred,	one hundred and one,	one hundred and two,	&c.

so likewise of succeeding numbers; thus, we have

345,	586:
three hundred and forty-five,	five hundred and eighty-six:

and again, 999 will be *nine hundred and ninety-nine,* which is the largest number capable of being expressed by *three* figures.

Here, the *first* figure on the right hand is said to occupy the *units' place,* the *second* the place of *tens,* and the *third* that of *hundreds.*

Of the auxiliary digit 0, the sole use is in the effect specified in the last two Articles; and all figures to the *right* of it will therefore be unaffected by it.

9. Def. 5. In estimating numerical magnitudes, we proceed in order from *hundreds,* to *thousands, tens of thousands,* and *hundreds of thousands; millions, tens of millions,* and *hundreds of millions;* in precisely the same manner as we have done above from *units* to *tens,* and from *tens* to *hundreds.*

10. Def. 6. Agreeably to the principle of Article (7), it is *assumed* that "*any* figure placed on the right of one or more others, has the effect of increasing every one of them *tenfold* without altering its own value;" and this enables us to express with facility any number whatever.

Thus,

(1) 1000 will represent One *Thousand.*

(2) 5493 will represent Five *Thousands,* four *Hundreds,* and ninety-three.

(3) 23456 will represent Twenty-three *Thousands*, four *Hundreds*, and fifty-six.

(4) 729054 will represent Seven hundred and twenty-nine *Thousands*, and fifty-four.

(5) 1803205 will represent One *Million*, eight hundred and three *Thousands*, two *Hundreds*, and five.

(6) 32754081 will represent Thirty-two *Millions*, seven hundred and fifty-four *Thousands*, and eighty-one.

(7) 473025004 will represent Four Hundred and seventy-three *Millions*, twenty-five *Thousands*, and four.

11. If the first three figures beginning from the right hand be denominated so many *units*, tens of *units* and hundreds of *units*, it follows that the next three figures taken the same way will be *thousands*, tens of *thousands*, and hundreds of *thousands:* the next three in order will be *millions*, tens of *millions*, and hundreds of *millions:* and so on.

Whence, to express in figures any number proposed, we have only to consider in which of these divisions each part of it ought to be found, observing that *three* figures from the right must be taken to make each division *complete*, before we proceed to the next.

Ex. 1. Express by means of figures; *Thirty-five thousand, eight hundred and nineteen.*

Here, eight hundred and nineteen belongs to the *first* division on the right, and is written 819:

also, thirty-five thousand must be found in the *second* division from the right, and is 35:

whence, the proposed number will be expressed by

3 5 8 1 9.

Ex. 2. Write down in figures the number; *Five million, twenty-five thousand, six hundred and seven.*

In this case, the first division on the right will be 607; the second will be 025, the digit 0 being affixed to the left of the others without altering their values, to make up the required number of *three;* and the third is 5: so that the expression required will be

5 0 2 5 6 0 7.

Ex. 3. Express by figures the following number;

Five hundred and seventy million, two hundred and six thousand and fifty-four.

Here, the first division is 054, the 0 altering only the values of the figures in the *subsequent* divisions: the second division is 206, and the third is 570: whence the number proposed is correctly expressed by

570206054.

12. This method of notation can never present any difficulty, provided it be carefully remembered that every division of figures as we proceed from the right hand towards the left must be *completed* as far as it is possible; and by a little practice, we shall be enabled to write down any number by beginning at the *left* hand.

Ex. To write down *Six hundred and thirteen million, five hundred and nineteen,* we observe that the division of millions will be 613: that of thousands will be 000, and that of units 519: so that the number is expressed in arithmetical symbols by

613000519.

13. *Examples for Practice in Notation.*

(1) Five hundred and ninety-eight.

(2) Seven thousand, eight hundred and four.

(3) Eighty-nine thousand and sixty-three.

(4) Six hundred and three thousand, two hundred and forty.

(5) Nine million, forty-three thousand, six hundred and two.

(6) Forty-five million, three hundred and eighty-seven thousand and twenty-five.

(7) Three hundred and forty-nine million, four thousand and sixty-five.

(8) One hundred million, ten thousand and one.

(9) Eight hundred and forty-two million, two hundred and forty-eight thousand, four hundred and eighty-four.

(10) Nine hundred and nine million, nine thousand and ninety-nine.

14. As far as practical utility is concerned, we shall seldom or never have occasion to express by figures,

numbers exceeding *Hundreds of Millions;* but the system of Notation admits of being extended so as to represent any number whatever.

Thus, instead of supposing that each division consists of *three* figures, if we include *six* figures as far as we can in each division from the right hand, the first may be regarded as so many hundreds of thousands of *Units;* the next as so many hundreds of thousands of *Millions;* the next as so many hundreds of thousands of what are called *Billions*, and the succeeding divisions, of so many hundreds of thousands of what are termed *Trillions*, *Quadrillions*, &c.

Ex. To represent *Ten thousand millions* by figures; for the first division we have 000000, and for the second 10000, so that the representation required is

1 0 0 0 0 0 0 0 0 0 0.

15. It will be observed, from what has been said, that each of the nine figures or digits,

1, 2, 3, 4, 5, 6, 7, 8, 9,

has an *absolute* value of itself, whereas the auxiliary digit 0 has no such value; and on this account the former are termed *significant* figures, in contradistinction to the last. It will moreover have occurred to the reader, that every one of these significant digits, in addition to its *absolute* value, which is fixed and certain, possesses also a *local* value dependent upon the situation in which it is placed; thus, in the expression of the number,

Four thousand three hundred and twenty-one,

which will be

4 3 2 1,

the 1 in the first place on the right hand, retains its absolute value; the second figure 2, in its situation denotes two *tens* or *twenty*; the third is three *hundred*, and the fourth is four *thousand;* so that the local values of 2, 3, and 4 here, are respectively, *ten* times, a *hundred* times and a *thousand* times, as great as their absolute values: and it is the circumstance of assigning to each of the significant figures a *local* as well as an *absolute* value, which confers upon the system, the immense powers it possesses.

NUMERATION.

16. Def. ' *Numeration* is the art of reading or estimating the value of a number expressed by figures, and is therefore the *reverse* of Notation.

17. From the circumstance of every figure possessing a local as well as an absolute value, it follows that the value of each figure must be estimated by the place which it occupies: hence, a figure standing by itself expresses so many *units;* a figure in the second place from the right, denotes so many *tens;* a figure in the third place, so many *hundreds,* and so on: consequently, if we suppose any numerical expression to be divided into *periods*, or portions each consisting of three figures as far as they go, the figures of the period on the right will be *units,* and tens and hundreds of *units;* those of the next will be units, tens and hundreds of *thousands*; those of the third will be units, tens and hundreds of *millions*; and so on.

Thus,

(1) 25 is Twenty-five.

(2) 304 is Three hundred and four.

(3) 5287 is Five thousand, two hundred and eighty-seven.

(4) 60539 is Sixty thousand, five hundred and thirty-nine.

(5) 207385 is Two hundred and seven thousand, three hundred and eighty-five.

(6) 1739204 is One million, seven hundred and thirty-nine thousand, two hundred and four.

(7) 35024376 is Thirty-five million, twenty-four thousand, three hundred and seventy-six.

(8) 275008005 is Two hundred and seventy-five million, eight thousand and five.

In each of these instances we conceive the expression to be separated into periods of three figures each as far as they go, beginning at the right hand: as in 275008005, we observe that 005 is the first period, 008 the second, and the third period is 275, each consisting of three figures: that is, 275 denotes two hundred

and seventy-five millions, 008 eight thousands, and 005 five units.

18. The last Article will be rendered more clear by the following scheme, called the *Numeration Table*:

&c, &c.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
	9	8	7	6	5	4	3	2	1
		9	8	7	6	5	4	3	2
			9	8	7	6	5	4	3
				9	8	7	6	5	4
					9	8	7	6	5
						9	8	7	6
							9	8	7
								9	8
									9

wherein the local value of every figure in each of the horizontal rows is pointed out by the name written *upwards* at the top of the whole: thus, in the *third* horizontal line from the bottom, the figures will be read *Nine hundred and eighty-seven;* and in the *second* line from the top, *Ninety-eight million, seven hundred and sixty-five thousand, four hundred and thirty-two.*

19. *Examples for Practice in Numeration.*

(1) 4320.	(7) 20084216.
(2) 87054.	(8) 79030284.
(3) 903756.	(9) 321408653.
(4) 2714325.	(10) 408076032.
(5) 8047328.	(11) 314159265.
(6) 12870045.	(12) 571268405.

20. We now proceed to the consideration of the *four* fundamental Operations that can be performed upon numbers, which are those of *Addition, Subtraction, Multiplication* and *Division*, each of which will be defined, explained and exemplified in its order.

I. ADDITION.

21. Def. *Addition* consists in finding a number equal to two or more numbers taken together, and this number is called their *Sum*.

Ex. 1. To find the sum of 2, 5 and 9; we see that *two* units and *five* units taken together are *seven* units, and these with *nine* units more, will amount to *sixteen* units, which is written 16: and we have the following *form*:

$$\begin{array}{r} 2 \\ 5 \\ \underline{9} \\ \underline{16} \end{array}$$

Ex. 2. Add together the numbers 254, 893 and 487.

It would be absurd to collect into one sum, numbers of different local values, as for instance, to say that three *units* and five *tens* amount to either eight *units* or eight *tens*; and we therefore place the numbers to be added together in such a *form* that each of the figures of the same denomination may be in the same *vertical* line, as on the *left* of the page:

Common Form.	*Explanation of Operation.*				
2 5 4	2 0 0	and	5 0	and	4
8 9 3	8 0 0	...	9 0	...	3
4 8 7	4 0 0	...	8 0	...	7
2 1	1 4 0 0		2 2 0		1 4
1 6 3 4	2 0 0		1 0		
	1 6 0 0		2 3 0		

and then, as is seen in the operation on the *right*, we have first added the *units* together and thus have 14 units, or 1 ten and 4 units; we have next found the sum of the *tens* to be 22, which with the 1 ten before obtained amount to 23 tens, or to 2 hundreds and 3 tens; and lastly, we have obtained 14 *hundreds*, which together with the 2 hundreds just found make 16 hundreds, or 1 thousand and 6 hundreds: whence, the entire

sum is 1 thousand, 6 hundreds, 3 tens and 4 units, or 1634.

The reasoning here used is thus applied to the figures on the left of the page: the numbers of tens and hundreds found by adding the vertical columns of units and tens are *annexed*, or *carried* to the columns of tens and hundreds respectively, and they are *here* put down *under* them just above the horizontal line; but in *practice* they are *always* omitted, by *mentally* adding them to the lowest figures of the next vertical rows.

22. To effect the operation of Addition, as appears from these two instances, it is merely necessary to know from *memory* or by *practice*, the sums of every two single figures: and the reasoning above employed leads to a general conclusion which is comprised in the following Rule.

Rule for performing Addition.

Place the numbers under one another in such a manner that units may stand under units, tens under tens, hundreds under hundreds, and so on, and draw a line below all the horizontal rows of figures: then add up the figures in the first vertical row on the right hand, find the numbers of *tens* and *units* in their sum, and put down the number of *units*, whether it be zero or any of the nine other digits: *carry* as many *units* as there are *tens* thus found to the next vertical row, and add them up as before, observing the numbers of *tens* and *units* contained in the sum: place the number of *units* under the row added, and carry the number of *tens* to the next; proceed in the same manner till the last row is added, when put down the numbers both of *tens* and *units*, as there are no more figures of higher denominations.

23. To ascertain whether the operation is correctly performed, various expedients might be resorted to; as for instance, that of adding the numbers *downwards* instead of *upwards*, which because the *same* set of numbers cannot have two *different* sums, must give the same result: but the only one, with this exception, which does not involve principles *hereafter* to be explained, seems to be that of omitting one of the horizontal rows of figures in a *second* operation, and afterwards adding it to the result of the rest obtained by the rule.

Addition.	*Proof.*
9358	4162
4162	8920
8920	6328
6328	19410
28768	9358
	28768

Here, 28768 is the sum: and omitting the *first* horizontal row, we find the sum of the *rest* to be 19410, to which the row 9358 being now added, produces 28768 the entire sum: whence we infer with some degree of probability, that the addition is correct: and this probability may be further increased by repeating the operation, with the omission of *any* horizontal row of figures.

24. *Examples for Practice in Addition.*

(1)	(2)	(3)	(4)
90	347	7153	29051
45	238	2857	73826
73	410	4105	57295

(5)	(6)	(7)	(8)
84	293	4028	53296
7	75	354	109
29	409	95	5875
13	3	2076	24658

(9)	(10)	(11)	(12)
73	235	736	25385
24	97	400	90624
9	958	4159	87653
251	64	47	40706
48	186	7204	97341

(13) Add together 432, 8076, 458 and 5431.

(14) Add together 72853, 27621, 45760, 820547 and 71425.

(15) Add together 205087, 32471, 29185, 1475 and 273.

(16) Find the sum of 72638594, 27836, 7805, 5271

and 1468357: and prove it to be correct by the omission of *each* horizontal row in succession.

(17) Find the sum of *Twenty-five million and four; Forty-seven thousand, two hundred and nine; Three hundred million, ten thousand and one; Sixty-five thousand and eighty-seven*, and *Five million and fifty:* write it down in words; and apply the ordinary proof of its being correct.

25. Def. It is usual, in the applications of Arithmetic, to express the operation of *Addition* by *signs* invented for the purpose: thus, the sum of 4 and 5 is expressed in the *form,*

$$4 + 5 = 9,$$

wherein the sign + between 4 and 5 denotes the addition of the latter number to the former, and is read *plus*, or *more by;* and the sign = between 5 and 9 expresses the result of such addition to be 9, or the *equality* between the *sum* of the numbers 4 and 5 and the *number* 9: so that the arithmetical *expression*

$$4 + 5 = 9,$$

is *read*

4 *plus* 5 *equals* 9.

Similarly, $2 + 3 + 7 = 5 + 7 = 12$, shews the sum of the three numbers 2, 3, 7, to be 12. So, in Ex. 2, of Article (21), we have $254 + 893 + 487 = 1634$, *expressive* of the operation there *performed.*

II. SUBTRACTION.

26. Def. *Subtraction* consists in finding *how much* one number exceeds another, and the excess is styled the *Remainder* or *Difference.* The greater of the proposed numbers is called the *Minuend*, and the less the *Subtrahend.*

Ex. 1. Find the difference of 7 and 2.

Here, it is evident that 7 units being equal to 2 units and 5 units taken together, if we *withdraw* or *subtract* the former, we shall have 5 units for the remainder or difference; and the operation is written in the *form:*

$$\begin{array}{c} 7 \\ \underline{2} \\ \underline{5} \end{array}$$

Ex. 2. To subtract 19 from 37, we place the figures as in the last example, and have

Common Form.	*Explanation of Operation.*
37	20 and 17
19	10 ... 9
18	10 ... 8

Here, the figure in the units' place of the upper line being *less* than that in the lower, it is impossible to subtract the lower from the upper: but by considering, as on the right of the page, the 7 as 17 by taking one of the units from the 3, we find the excess of 17 above 9 to be 8, which is put in the units' place of the remainder, and then we have to take away 1 from 2 instead of 3, in consequence of having regarded the 7 as 17: hence the remainder in the tens' place will be 1, and the difference of the two numbers is 18.

The figure in the lower line being greater than that in the upper, we have *borrowed* ten units of the *next* denomination; but the same result is obtained whether we suppose 1 to be *subtracted* from the *upper* line, or *added* to the *lower*, as the remainder will evidently be the same on both suppositions. In *practice* we add ten units of any denomination to both the quantities concerned; to the upper as *ten* of that denomination, and to the lower as *one* of the next superior denomination, and by this contrivance the remainder is unaffected.

27. Hence it appears to be necessary to *recollect* the differences of every two numbers less than 20: and the reasoning being applicable to all other instances, the result of it may be embodied in the following Rule.

Rule for performing Subtraction.

Place the less number under the greater, so that units may stand under units, tens under tens, and so on; begin at the units' place and subtract each figure in the lower line from that in the upper, taken by itself, or increased by 10, according as it is greater or less than the said figure in the lower line, and put down the remainder; observing that whenever *ten* units have been *borrowed*, or added to the upper line, *one* unit must be

carried, or added to the next denomination in the lower line.

28. Subtraction being the *reverse* of Addition, it follows, that if we add together the remainder and the less of the numbers proposed, the sum ought to be equal to the greater; and the operation of subtraction may be *presumed* to be correct when this is the case: thus,

Subtraction.	*Proof.*
9 6 2 8 the Minuend:	6 7 5 9 the Subtrahend:
6 7 5 9 the Subtrahend:	2 8 6 9 the Remainder:
2 8 6 9 the Remainder:	9 6 2 8 the Minuend:

where the last result is the same as the greater of the numbers proposed; and thence we infer that the operation has been correctly performed.

29. *Examples for Practice in Subtraction.*

(1)	(2)	(3)	(4)
1 4	7 9	4 2 8	7 0 4 6
8	4 5	2 7 4	8 0 7

(5)	(6)	(7)
6 2 8 3 1	5 4 2 6 5 7	2 0 4 0 8 7
4 8 0 7 2	2 1 4 9 5 8	7 6 4 9 8

(8) What is the excess of 12795 above 8096?

(9) From 9261374 take 2548298.

(10) Find the difference of 20470932 and 80476325.

(11) How much greater is 12785462 than 1842567?

(12) Required the excess of *Three hundred and five million, two hundred and four,* above *Seventy-five thousand, three hundred and eighty-six.*

30. Def. The operation of *Subtraction,* is indicated or expressed by the sign $-$, which is read *minus,* or *less by,* with the use of the sign $=$; thus, the excess of 7 above 3, will be expressed in the *form,*

$$7-3=4,$$

which is *read*

7 *minus* 3 *equals* 4:

where the sign – between 7 and 3 denotes the subtraction of the latter from the former, and the sign = between 3 and 4 shows the *equality* of the excess to 4.

III. MULTIPLICATION.

31. DEF. *Multiplication* consists in finding the *amount* of a number, when *repeated* any number of times, and this amount is termed the *Product.* The former of these numbers is called the *Multiplicand,* and the latter the *Multiplier.*

Ex. 1. To multiply 7 and 42 by 4 and 5 respectively, being to find the sums arising from the numbers 7 and 42, *four* and *five* times repeated, we may *determine* the products as underneath;

$$\begin{array}{r|r} & 42 \\ 7 & 42 \\ 7 & 42 \\ 7 & 42 \\ 7 & 42 \\ \hline 28 & 210 \end{array}$$

but the operations are *expressed* more briefly, as follows:

$$\begin{array}{r|r} 7 & 42 \\ 4 & 5 \\ \hline 28 & 210 \end{array}$$

Ex. 2. Find the products arising from the multiplication of 256 by 10, 11 and 12 respectively.

By Article (10) we know that 256 will become *ten* times as great by merely affixing to the right of it the auxiliary digit 0, and thus we have the following operation:

$$\begin{array}{rl} 2\ 5\ 6 & \text{the multiplicand:} \\ 1\ 0 & \text{the multiplier:} \\ \hline 2\ 5\ 6\ 0 & \text{the product.} \end{array}$$

To multiply 256 by 11, we consider that 11 being equal to 1 and 10 together, the product will be equal to the sum of 256 taken *one* and *ten* times: thus,

$$\begin{array}{rl}
256 & \\
11 & \\
\hline
256 & = 256 \text{ taken } \textit{once}, \\
2560 & = 256 \text{ taken } \textit{ten} \text{ times}, \\
\hline
2816 & = 256 \text{ taken } \textit{eleven} \text{ times}:
\end{array}$$

that is, 2816 is the product of 256 by 11: and the omission of the 0 on the right of the *fourth* line in the operation, can cause no inconvenience, as the *places* of the succeeding figures adequately determine their values.

To find the product of 256 by 12, 256 must be taken *twice* and *ten* times together, and we have

$$\begin{array}{rl}
256 & \\
12 & \\
\\
\left.\begin{array}{r} 256 \\ 256 \end{array}\right\} & = 256 \text{ taken } \textit{twice}, \\
2560 & = 256 \text{ taken } \textit{ten} \text{ times}, \\
\hline
3072 & = 256 \text{ taken } \textit{twelve} \text{ times}:
\end{array}$$

whence, the product of 256 by 12 is 3072, the observation above made holding good with respect to the omission of the 0 at the end of the *fifth* line.

32. From the mode in which the results above have been obtained, it is manifest that Multiplication is merely a *compendious* method of performing the addition of two or more *equal* numbers: and the following scheme, which is termed the *Multiplication Table*, presents at one view the product arising from the multiplication of any two numbers not exceeding 12; and though the products of the nine *digits* form the *basis* of those of all numbers whatever, it is here extended for the sake of *practical* convenience, and should be carefully committed to memory.

THE MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

In this table, the first horizontal line consists of the first twelve numbers in order: the second consists of the products of the same numbers when multiplied by 2: the third contains their products when multiplied by 3: the fourth when multiplied by 4, and so on: and the table is repeated in the following manner.

Thus, to make use of the *second* line of figures, we say

twice 1 are 2,	twice 5 are 10,	twice 9 are 18,
twice 2 are 4,	twice 6 are 12,	twice 10 are 20,
twice 3 are 6,	twice 7 are 14,	twice 11 are 22,
twice 4 are 8,	twice 8 are 16,	twice 12 are 24.

Ex. 1. Let it be required to multiply 845 by 6:

then, since the product of 854 by 6 is evidently equal to the sum of the products of all its parts, namely, 800 and 50 and 4, by 6, we have

$$\begin{array}{rl} 854 & \\ 6 & \\ \hline 24 & = \text{product of } 4 \text{ by } 6: \\ 300 & = \text{product of } 50 \text{ by } 6: \\ 4800 & = \text{product of } 800 \text{ by } 6: \\ \hline 5124 & = \text{product of } 854 \text{ by } 6. \end{array}$$

In practice, we *mentally* combine into one sum, the figures of all these products as they arise: thus, first multiplying 4 by 6, we find the product to be 24 by the table; and having placed the 4 *units* under those of the quantity proposed, we carry the 2 *tens* to the product of 5 by 6, which is here 30 *tens*, and thus obtain 32 tens; whereof the 2 being put under the *tens'* place, and the 3 being carried to the product of 8 by 6, or to 48 *hundreds*, the entire number of hundreds is 51; and the whole product is 5124.

Ex. 2. Multiply 486 by 357.

Here, proceeding with each of the figures 7, 5 and 3, according to the last example, we have

$$\begin{array}{rl} 486 & \\ 357 & \\ 3402 & = \text{product of } 486 \text{ by } 7: \\ 2430 & = \text{product of } 486 \text{ by } 50: \\ 1458 & = \text{product of } 486 \text{ by } 300: \\ \hline 173502 & = \text{product of } 486 \text{ by } 357. \end{array}$$

In this instance, the *situations* of the figures in the fourth and fifth lines render them equivalent to the products of 486 by 50 and 300 respectively, without supplying the auxiliary digits 0.

If one or more of the figures of the multiplier be 0, it is evident that the corresponding *partial* product will be 0, and the lines may be entirely omitted after

placing down each 0 *once,* to give the proper value to the product arising from the next figure.

33. The reasoning here employed being independent of the examples made use of to illustrate it, we are enabled to lay down a Rule in the following words.

Rule for performing Multiplication.

Place the multiplier under the multiplicand, as before, and draw a line under the whole: multiply *every* figure in the multiplicand by the figure in the units' place of the multiplier, observing to carry to the next product the number of *tens* in that arising from the multiplication of any of the digits in the multiplicand, and to place down the *units* under the figure multiplied, till the last product is obtained, which place down in *full:* proceed in the same manner with the figure of the multiplier in the tens' place, the figure on the right of this product being placed under the said figure; then with the figures in the succeeding places; *add* these products together, and the sum will be the entire product.

34. If the multiplicand and multiplier change places, the product must be the same as before, otherwise the *same* numbers would have *more* products than *one;* and if the products be the *same,* we have some proof that the operation has been correctly performed in each case. Thus,

Multiplication.		*Proof.*	
875		427	
427		875	
6125	8	2135	8
1750	2 × 4	2989	4 ×
3500	8	3416	8
373625		373625	

The best *practical* proof of this operation by "*Casting out the Nines,*" depends upon that of the *next* subdivision: but we will enunciate the Rule, and apply it to this example, in the *form* usually adopted. "Find the *sums* of the *figures* in the *Multiplicand* and *Multiplier,*

and determine the *Remainders* when divided by 9: then, if the remainder arising from dividing the product of these *two* remainders by 9, be the same as that which arises from dividing the sum of the digits in the *final* result by 9, the operation has *probably* been correctly performed."

Thus, in the instance above, the sum of the figures in 875 is 20, and the remainder is 2: in 427, the sum of the figures is 13, and the remainder is 4; therefore the remainder arising from dividing the product of 2 and 4 by 9 is 8; and there is also a remainder 8, if we divide the *sum* of the digits comprised in 373625 by 9.

The remainders are *placed* as above, and the reason of the rule will be given in the *Appendix*.

Abbreviations of Multiplication.

35. The ordinary process of Multiplication may be *shortened* or *facilitated*, as in the following instances.

Ex. 1. Multiply 257 by 6400, and 790 by 8300.

Here, omitting the ciphers on the right, or *supposing* them to be omitted, we have

```
    2 5 7            |      7 9 0
      6 4 0 0        |      8 3 0 0
    -------          |    -------
    1 0 2 8          |    2 3 7
  1 5 4 2            |  6 3 2
  -------------      |  -------------
  1 6 4 4 8 0 0      |  6 5 5 7 0 0 0
```

where the ciphers are *annexed* at last to the right of the products obtained in the ordinary way, to give the other figures their proper local values.

Ex. 2. Required the product of 537 by 63.

Here, 63 being the product of 7 and 9, it follows that 7 times any number 9 times repeated, is the same as 63 times that number: whence, we have

```
    5 3 7
        7 times 9=63
  -------
  3 7 5 9
        9
  -------
3 3 8 3 1 =the product.
```

To multiply 476 by 47, we have these operations:

476	476
9	8
4284	3808
5	6
21420	22848
952	476
22372	22372

in the former of which *twice* 476 is *added,* and in the latter *once* 476 is *subtracted,* in order that 476 may be repeated 47 times, *exactly.*

36. The same mode of reasoning and similar operations may be used to find the product of *more* than two numbers, which is called the *Continued Product* of so many *Factors.*

Ex. To find the product of 3, 5 and 47,

multiply 3
by 5
15 is the product:

again, multiply 15
by 47
105
60
705 is the product: that is,

705 is the continued product of the factors 3, 5 and 47.

37. *Examples for Practice in Multiplication.*

(1) 284	(2) 1475	(3) 2867
2	3	4
(4) 78543	(5) 41087	(6) 942763
5	6	7

(7) $\begin{array}{r} 8536274 \\ 8 \\ \hline \end{array}$ (8) $\begin{array}{r} 3216795 \\ 9 \\ \hline \end{array}$ (9) $\begin{array}{r} 1468725 \\ 11 \\ \hline \end{array}$

(10) $\begin{array}{r} 6283195 \\ 12 \\ \hline \end{array}$ (11) $\begin{array}{r} 21584 \\ 17 \\ \hline \end{array}$ (12) $\begin{array}{r} 39265 \\ 39 \\ \hline \end{array}$

(13) $\begin{array}{r} 921846 \\ 158 \\ \hline \end{array}$ (14) $\begin{array}{r} 827941 \\ 376 \\ \hline \end{array}$ (15) $\begin{array}{r} 5086927 \\ 495 \\ \hline \end{array}$

(16) $\begin{array}{r} 279420 \\ 7350 \\ \hline \end{array}$ (17) $\begin{array}{r} 254037 \\ 2980 \\ \hline \end{array}$ (18) $\begin{array}{r} 4785328 \\ 7802 \\ \hline \end{array}$

(19) Multiply 123456789 by each of the numbers 2, 3, 4, 5, 6, 7, 8 and 9.

(20) Find the product of 47691 and 27: of 28573 and 35: of 716281 and 48: of 129385 and 66: of 138476 and 81: of 480765 and 97, and of 8241763 and 123.

(21) Required the continued product of 4, 7 and 25: of 13, 15 and 17: and of 35, 29, 43 and 87.

38. Def. The operation of *Multiplication* is expressed by the sign ×, which is read *into*, or *multiplied by*: thus,

$$5 \times 7 = 35$$

denotes the result of the multiplication of 5 by 7 to be 35:

so, again, $4 \times 5 \times 13$ expresses the continued product of 4, 5 and 13, which

$$= 20 \times 13 = 260:$$

and we have

$$(8+3) \times (7-2) = 55,$$

expressive of the product of the *sum* of 8 and 3, and the *difference* of 7 and 2, which may be more briefly written

$$11 \times 5 = 55.$$

IV. DIVISION.

39. Def. *Division* consists in finding how *many times* one number is contained in another, and the number of such times is termed the *Quotient*. The former

of these numbers is called the *Divisor,* and the latter the *Dividend.*

Ex. 1. To divide 6 by 2, and 219 by 52 respectively, we must obviously take the latter numbers from the former in each case, as often as we are able, according to the principle of Subtraction before explained: thus,

$$\begin{array}{r} 6 \\ 2 \\ \hline 4 \\ 2 \\ \hline 2 \\ 2 \\ \hline 0 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 219 \\ 52 \\ \hline 167 \\ 52 \\ \hline 115 \\ 52 \\ \hline 63 \\ 52 \\ \hline 11 \\ \hline \end{array}$$

so that after *three* subtractions in the former case, there is *no* remainder, whereas in the latter, *four* such operations leave a remainder 11: that is, 2 is contained in 6, *three* times *exactly;* but 219 divided by 52 gives 4 for the *quotient,* with 11 for the *remainder:* and the processes are made to take the following *forms:*

$$\begin{array}{r} 2\,)\,6 \\ \hline 3 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 52\,)\,219\,(4 \\ 208 \\ \hline 11 \\ \hline \end{array}$$

From these instances of what are called *Short* and *Long* division, it follows that Division is the *reverse* of Multiplication: and hence, by a reversed process, the Multiplication Table must furnish the means of obtaining the quotient.

Ex. 2. If we multiply 349 by 215, the product is 75035: and therefore, the quotient of 75035 when divided by 349 must be 215, which will be found by reversing the operation as follows:

```
349)75035(215
    698      =product of 349 by 2 denoting 200 units:
     523
     349     =product of 349 by 1 denoting 10 units:
     1745
     1745    =product of 349 by 5 denoting 5 units.
```

Here, the first figure 2 in the quotient is obtained by inquiring how often 3 is contained in 7, or 34 in 75: then, after multiplying 349 by 2, which, from the places of the figures, represents 2 *hundreds*, and subtracting the product which is 698, from 750, we have a remainder 52: to this the next figure 3 of the dividend being *annexed*, we seek how often 3 is contained in 5, or 34 in 52, and this quotient being 1, 1 *ten* is annexed to the 2 *hundreds* already obtained: multiplying and subtracting as before, we bring down the last figure 5 of the dividend, and find the corresponding quotient to be 5 *units* exactly; and the operation is then completed, leaving no remainder.

In this example, the dividend has *virtually* been broken up into *parts* each exactly divisible by 349, as will appear by supplying the auxiliary digits in the form of *Long Division*: thus,

```
Divisor.        Dividend.             Quotient.
 349)69800+3490+1745(200+10+5
     69800
          +3490
           3490
                +1745
                 1745
```

or, in the form of *Short Division*, as below:

```
Divisor.      Dividend.
349)69800+3490+1745
     200 +  10 +  5  Quotient.
```

40. The principles of the reasoning here employed may be embodied in the following general Rule.

Rule for performing Division.

Place the divisor and dividend in the same line, separated by a small curved line; and on the right of the dividend draw another line of the same kind; inquire how often the first one or two figures on the left hand of the divisor are contained in the first one or more of those of the dividend, and place the result on the right as the first figure of the quotient: and the product arising from the multiplication of the divisor by this figure being subtracted from the dividend, *bring down* or *annex* to the remainder the next figure of the dividend and let the same kind of operation be repeated till every figure of the dividend is disposed of; then the quotient, and the remainder if any, will be ascertained.

If the divisor do not exceed 12, these operations may be performed *mentally*, the quotient and remainder being placed in a line immediately under the dividend.

41. Since the quotient is the result arising from the division of the dividend by the divisor, it follows that the dividend must be the product arising from the multiplication of the divisor by the quotient, or of the quotient by the divisor: also, if there be any remainder, it must evidently be added to this product to produce the true dividend, since the whole is equal to the sum of all its parts; and hence we have a method of shewing whether the division has been correctly performed, or not.

Ex. Find the quotient and remainder when 275487 is divided by 736.

Division.

```
736)275487(374
    2208
    ----
     5468
     5152
     ----
      3167
      2944
      ----
       223
```

Proof.

```
   374
   736
  ----
  2244
 1122
2618
------
275264
   223
------
275487
```

The *easiest* proof of this operation, is that of adding together the figures of the remainder and the *partial* products of the divisor in vertical lines, since the sum thus formed ought manifestly to be equal to the dividend when the work is *right*, as in the following *form* which is *omitted* in practice. See the *Appendix.*

```
2208
 5152
  2944
   223
------
275487  the dividend.
```

Abbreviations of Division.

42. The operation of Division may, in particular cases, be made to comprise *fewer figures*, or to take up *less room*, by such considerations as follow.

Ex. 1. To divide 20573290 by 34500, we have

```
Divisor.         Dividend.     Quotient.
345,00)205732,90(596
       1725
       ----
        3323
        3105
        ----
         2182
         2070
         ----
          11290 remainder;
```

where after the two *ciphers* in the divisor and the two *figures* 90 in the dividend are *cut off*, the operation is effected by the ordinary method, the said two figures of the dividend being *annexed* to the remainder at last, inasmuch as 112 from the places of the figures is equivalent to 11200.

Ex. 2. Divide 792415 by 72.

Here, since 72 is the product of 8 and 9, it is obvious, from Ex. (2), of Article (35), that the quotient may be obtained from *successive* divisions by 8 and 9:

$$\begin{array}{l} 8\,)\,7\,9\,2\,4\,1\,5 \\ 72 \quad 9\,)\ \ \underline{9\,9\,0\,5\,1},\ 7 \text{ first remainder:} \\ 1\,1\,0\,0\,5,\ 6 \text{ second remainder:} \end{array}$$

and we have now only to deduce the *true* remainder from the *two* remainders just found.

The dividend at first being so many *units*, the first remainder 7 must be *units;* but the second dividend being the result of the division by 8, must be regarded as so *many times* 8, and the second remainder will therefore be 6 times 8, or 48 *units:* whence, the *true* remainder will be

$$6 \times 8 + 7 = 55 \textit{ units:}$$

and we may lay down a rule in the following words.

In dividing by *two* numbers, instead of *one* equal to their product, the *true remainder* is equal to the product of the *last remainder* and the *first divisor*, together with the *first remainder.*

43. *Examples for Practice in Division.*

(1) 2)348 (2) 3)4596 (3) 4)276284

(4) 5)84375 (5) 6)53844 (6) 7)536074

(7) 8)958324 17 (8) 9)7163253651

(9) 10)3158367 (10) 11)1234567890

(11) 12)9876543 (12) 23)14415724 6(

(13) 37)47073256((14) 549)48310567(

(15) 7038)140167329((16) 7900)25413286(

(17) 5730)8327970((18) 1480)64157600(

(19) Find the quotient of 76294 by 32: of 729518 by 49: of 8015473 by 66, and of 4050873 by 121; and prove the correctness of the operations.

44. The operation of *Division* is expressed by means of the sign $\div$ and sometimes :, which is read *by*, or *divided by*; thus,

$$42 \div 7 = 6$$

implies that the result of the division of 42 by 7 is 6: again, $(70-7) \div (4+5)$ is equivalent to $63 \div 9 = 7$.

MEASURES AND MULTIPLES.

45. DEF. 1. A *Measure* of a number is *any* number which divides it without a remainder; as 4 is a measure of 24, because it is contained exactly 6 times in 24.

It is said to *measure* the number by the *units* contained in the *quotient*. All numbers have 1 for a measure; those, whereof 2 is a measure, are called *even* numbers, admitting of being divided into two *equal* parts; and all other numbers are termed *odd* numbers.

46. DEF. 2. A *Common Measure* of two or more numbers is *any* number, which will divide each of them without leaving a remainder; and the greatest of such measures is called the *Greatest Common Measure*, or *Greatest Common Divisor*: thus, 3 is *a* common measure of 18 and 30; whereas 6 is their *greatest* common measure, being the greatest number capable of dividing each of them without a remainder.

47. DEF. 3. An *Aliquot Part* of a number is *any* measure of it.

48. DEF. 4. A *Multiple* of a number is *any* number which is divisible by it, or contains it an exact *number of times*; as 108 is a multiple of 12, because 12 is contained exactly 9 times in 108.

49. DEF. 5. A *Common Multiple* of two or more numbers is *any* number which is divisible by each of them separately; and the *Least Common Multiple* is the least number that can be divided by each of them without a remainder: as 24 is *a* common multiple of 3 and 4, because divisible by both of them; whereas 12 is their *least* common multiple, because it is the least number that both 3 and 4 can divide without leaving a remainder.

50. DEF. 6. A *Composite Number* is one which arises from the multiplication of *two* or *more* other numbers termed *Factors*; and it is thus distinguished from

an *Incomposite* or *Prime Number*, which cannot so originate: as 22 is a composite number, because it is equal to the product of the factors 2 and 11; but 11 is an incomposite or prime number, because the multiplication of no two or more factors will produce it, *unity*, which is merely the element of number, being excepted.

51. *If one number measure each of two others, it will measure their sum and difference: also, any multiples of each, their sums and differences.*

Thus, 4 is a common measure of 20 and 12; and

$$\text{their sum} = 20 + 12 = 32 = 4 \times 8:$$
$$\text{their difference} = 20 - 12 = 8 = 4 \times 2:$$
$$\text{a multiple of } 20 = 20 \times 5 = 100 = 4 \times 25:$$
$$\text{a multiple of } 12 = 12 \times 7 = 84 = 4 \times 21:$$

each of which evidently comprises the number 4 as a measure or factor: and similarly of more numbers.

52. *To find the greatest common measure of two numbers.*

Let the numbers proposed be 63 and 168: then resolving each of them into its *prime* factors, we have

$$63 = 7 \times 9 = 7 \times 3 \times 3:$$
$$168 = 7 \times 24 = 7 \times 3 \times 8 = 7 \times 3 \times 2 \times 2 \times 2:$$

and the greatest common measure is evidently 7×3 or 21, because 3 and $2 \times 2 \times 2$ or 8 have no common factor: and employing the principles of the last Article, we obtain the same result by the following *form:*

```
63) 168 (2
    126
    ---
     42) 63 (1
         42
         --
         21) 42 (2
             42
             --
```

where 21 the last *Divisor* is the greatest common measure: and we have hence the following Rule.

Rule for finding the Greatest Common Measure.

Divide the greater of the numbers by the less, and then the divisor by the remainder: repeat this opera-

tion till there is no remainder, and the last divisor will be the greatest common measure.

To ascertain the greatest common measure of three or more numbers, find the greatest common measure of any two of them: then that of this greatest common measure and another of them: and so on to the last.

Examples.

(1) Required the greatest common measure of 428571 and 999999.

```
Here, 428571)999999(2
             857142
             ------
             142857)428571(3
                    428571
                    ------
```

therefore 142857, being the last *divisor*, is the greatest common measure.

(2) What is the greatest common measure of the numbers 12, 42 and 63?

Here, by *inspection*, 6 is the greatest common measure of 12 and 42; and to find that of 6 and 63, we have

```
6)63(10
  60
  --
   3)6(2
     6
     -
```

therefore, 3 is the common measure required.

(3) Determine the greatest factor common to 741, 1131, 1183 and 1989.

Proceeding by the directions of the rule, we have

```
741)1131(1
     741
     ---
     390)741(1
         390
         ---
         351)390(1
             351
             ---
              39)351(9
                 351
```

or, 39 is the greatest common measure of 741 and 1131: and to find that of 39 and 1183, the process will be

$$\begin{array}{l} 39)\,1183\,(30 \\ \quad\;\; \underline{117} \\ \qquad\quad 13)\,39\,(3 \\ \qquad\qquad\;\; \underline{39} \end{array}$$

or, 13 is the greatest common measure of 741, 1131 and 1183:
whence, 13 is the factor to be determined, since it also divides 1989 without a remainder.

In the second example, it is immaterial in what *order* the numbers are taken; and in the last instance it will be found that the number required is the greatest common measure of the common measures of every *two* of them that can be selected.

Examples for Practice.

(1) 9 and 24.
(2) 126 and 144.
(3) 3556 and 3444.
(4) 5187 and 5850.
(5) 6441 and 10283.
(6) 13667 and 14186.
(7) 43365 and 44688.
(8) 11050 and 35581.
(9) 109056 and 179712.
(10) 16, 24 and 140.
(11) 13338, 14136 and 15903.
(12) 204, 1190, 1445 and 2006.

53. The following remarks will be of service in making use of the last rule.

If the figure in the units' place be divisible by 2, the number is divisible by 2.

If the figures in the units' and tens' places be 4, or be divisible by 4, the number is divisible by 4.

If the figures in the units', tens' and hundreds' places be 8, the number is divisible by 8.

If the sum of all the figures be divisible by 3 or 9, the number is divisible by 3 or 9.

If the figure in the units' place be 5 or 0, the number is divisible by 5.

If the sums of the alternate figures beginning at

either end be equal, or one sum exceed the other by 11, or by any multiple of it, the number is divisible by 11.

54. *To find the least common multiple of two numbers.*

To find the least common multiple of 18 and 30, we observe that

$$18 = 6 \times 3 \text{ and } 30 = 6 \times 5,$$

so that the least number which contains them both exactly is evidently $6 \times 3 \times 5 = 90$, or the product of 18 and 30 divided by 6 their greatest common measure: and hence we have the following Rule.

Rule for finding the Least Common Multiple.

Multiply *either* of the numbers by the quotient arising from dividing the *other* by the greatest common measure, and the product will be their least common multiple.

If there be more than two numbers, proceed in the same way with the least common multiple of any two of them and the third: and so on, till they are all taken.

Examples.

(1) What is the least common multiple of 209 and 304?

Here, we have the operations below:

$$\begin{array}{llll}
209)\,304\,(1 & & & \\
\quad\;\; 209 & & & \\
\cline{1-1}
\quad\;\; 95)\,209\,(2 & & & \\
\qquad\qquad\; 190 & & & \\
\cline{1-1}
\qquad\qquad\;\; 19)\,95\,(5 & & & \\
\qquad\qquad\qquad\quad\; 95 & & & \\
\cline{1-1}
\end{array}$$

so that 19 is the greatest common measure of the numbers proposed: and the number required will therefore $= (209 \times 304) \div 19 = (209 \div 19) \times 304 = 11 \times 304$, or, $= (304 \times 209) \div 19 = (304 \div 19) \times 209 = 16 \times 209$: both of which being multiplied out, amount to 3344.

(2) Determine the least common multiple of 64, 250 and 432.

The greatest common measures of 64 and 250 is 2,

and their least common multiple is 8000: the greatest common measure of 8000 and 432 is 16, and the least common multiple of the three numbers proposed will therefore be 216000: and, as in the preceding Article, the *order* in which the numbers are taken will have no influence upon the result.

Examples for Practice.

(1) 12 and 27. (2) 289 and 323.
(3) 849 and 1132. (4) 3, 4 and 16.
(5) 24, 39 and 376. (6) 12, 15, 35 and 56.

55. When the least common multiple of several numbers is required, and no measures common to any two or more of them appear at *first sight* to exceed 12, the *easiest* method is to place the numbers in a row, and to divide such of them, as admit of it, by the *primes*, or prime numbers 2, 3, 5, 7, 11, *repeated* when it can be done, as *often* as possible: then, the product of all *these divisors* and the numbers in the last line, will be the least common multiple.

Thus, to find the least common multiple of 2, 3, 8, 9, 15, 21 and 35, we shall have the following *scheme:*

$$
\begin{array}{r|l}
2 & 2,\ 3,\ 8,\ 9,\ 15,\ 21,\ 35 \\ \hline
2 & 1,\ 3,\ 4,\ 9,\ 15,\ 21,\ 35 \\ \hline
3 & 1,\ 3,\ 2,\ 9,\ 15,\ 21,\ 35 \\ \hline
5 & 1,\ 1,\ 2,\ 3,\ \ 5,\ \ 7,\ 35 \\ \hline
7 & 1,\ 1,\ 2,\ 3,\ \ 1,\ \ 7,\ \ 7 \\ \hline
 & 1,\ 1,\ 2,\ 3,\ \ 1,\ \ 1,\ \ 1
\end{array}
$$

and the least common multiple is

$$2 \times 2 \times 3 \times 5 \times 7 \times 2 \times 3 = 2520.$$

This Rule is founded upon Article (51); very little attention is required to see the *reason* of it, and the following examples will furnish its *practice.*

Examples for Practice.

(1) 3, 5, 9. (2) 8, 9, 12, 18. (3) 6, 15, 27, 35.
(4) 3, 9, 7, 15, 28, 42. (5) 8, 18, 28, 36, 54, 72, 90.

General proofs of all that has been said here, may be found in the Author's *Elements of Algebra.*

CHAPTER II.

APPLICATION OF ARITHMETIC TO NUMERICAL MAGNITUDES OF VARIOUS DENOMINATIONS, NOT CONNECTED BY THE BASE OF THE COMMON SYSTEM OF NOTATION.

56. In the preceding Chapter we have considered only such *abstract* numbers as are formed by figures whose local values are always regulated by the same fixed number *ten:* but the rules given are easily extended to *concrete* magnitudes wherein the local values of the figures are connected by more numbers than one; as for instance, to *Pounds, Shillings, Pence* and *Farthings,* where *four* farthings are equivalent to *one* penny, which is the next higher denomination; *twelve* pence to *one* shilling, which is the next denomination in order; and *twenty* shillings to *one* pound: the *different* numbers 4, 12 and 20 connecting the denominations, in the same manner as the *fixed* number 10, was supposed to connect the denominations of Integers.

The processes employed in cases of this nature are *Reduction,* and the fundamental operations then called *Compound Addition, Compound Subtraction, Compound Multiplication* and *Compound Division,* each of which will be exemplified in order: and the *Tables* by means of which they are conducted, will be found at the beginning of the work.

REDUCTION.

57. Def. *Reduction* is the converting or changing of numerical quantities, from one or more *denominations* to one or more others, such that the *real* or *absolute* values shall remain unaltered: and its operations will evidently depend upon the principles already explained.

Ex. Reduce £25. 13*s*. 6$\frac{3}{4}$*d*. into farthings; and perform the *converse* operation.

The correctness of the following operations will be

manifest from the explanations annexed to their several steps, which are omitted as unnecessary in *practice*:

Direct Operation.

£. s. d.
25 . 13 . $6\frac{3}{4}$
20s. = 1 pound:
£. s. d.
513s. = 25 . 13 . 0.
12d. = 1 shilling:
6162d. = 25 . 13 . 6.
4f. = 1 penny:
24651f. = 25 . 13 . $6\frac{3}{4}$.

Converse Operation.

far.
1d. = 4f.) 24651
1s. = 12d.) 6162$\frac{3}{4}$
£1. = 20s.) 513 . 6
£25 . 13 . $6\frac{3}{4}$.

Here, the *denominations* are separated by a point as (.); and this is necessary to distinguish them from *ordinary* numbers, which do not require it, because their local values are all fixed and certain: and each of these operations may be regarded as a *proof* of the other.

58. The former process is called a *descending*, and the latter an *ascending* reduction, and they lead respectively to the following rules.

Rule I. To reduce quantities from *higher* to *lower* denominations, *multiply* the highest denomination by the number which connects it with the next inferior, and to the product add the number of the inferior denomination in the quantity proposed; and repeat this for each succeeding denomination till the required one is obtained.

Rule II. To reduce quantities from *lower* to *higher* denominations, *divide* them by the numbers which connect the different denominations in order, and annex the remainders at each step, so as to retain the denominations of the dividends from which they respectively arise.

Ex. How many half-crowns are there in £253. 9s. 10d.?

Here both the rules are requisite, and we have the following operations:

$$\begin{array}{rl} & \text{£.}\ \ s.\ \ d. \\ & 253 \,.\, 9 \,.\, 10 \\ & \quad 20 \\ \hline & 5069s. \\ & \quad 12 \\ \hline 1 \text{ half-crown} = 30d. \,) & 60838d. \\ \hline \text{half-crowns} & 2027 \,.\, 28d.: \end{array}$$

that is, the proposed sum is equivalent to 2027 half-crowns with 28*d.* or 2*s.* 4*d.* remaining; and this result is verified by reversing the process: thus,

$$\begin{array}{rl} & \textit{half-c.}\ \ d. \\ & 2027 \,.\, 28. \\ & \quad 30 \\ \hline 12\,) & 60838d. \\ \hline 20\,) & 5069 \,.\, 10d. \\ \hline & \text{£}253 \,.\, 9 \,.\, 10d. \end{array}$$

Examples for Practice.

(1) Reduce £71. 13*s.* 6½*d.* into farthings; and verify the result.

Answer: 68810 farthings.

(2) Find the number of farthings in 95 guineas 17*s.* 9¾*d.*: and conversely.

Answer: 96615 farthings.

(3) Reduce £295. 18*s.* 3¾*d.* to farthings; and prove the reduction.

Answer: 284079 farthings.

(4) Find the number of pounds, &c., in 415739 farthings; and prove the operation.

Answer: £433. 1*s.* 2¾*d.*

(5) Reduce 14cwt. 3qrs. 24lbs. into ounces; and shew that the result is probably correct.

Answer: 26816 ounces.

(6) Find the number of ounces in 11cwt. 2qrs. 17lbs. 15oz.; and confirm the operation.

Answer: 20895 ounces.

(7) What number of cwts., &c., are contained in 65437 drams? and verify the result.

Answer: 2cwt. 1qr. 3lbs. 9oz. 13drs.

(8) Reduce 3tons. 14cwt. 3qrs. 25lbs. 11oz. 9drs. into drams; and prove the result.

Answer: 2149817 drams.

(9) Find the number of poles contained in 15mi. 5fur. 31po.; and verify the result.

Answer: 5031 poles.

(10) In 1081080 inches, how many miles, &c.? and prove it.

Answer: 17 miles, 110 yards.

(11) Reduce 304935 feet to miles, &c.; and give the converse operation.

Answer: 57mi. 6fur. 5yds.

(12) What number of inches are equivalent to 512yds. 2ft. 9in.? and prove the result.

Answer: 18465 inches.

(13) Reduce 54yds. 8ft. 104in., superficial measure, into inches.

Answer: 71240 inches.

(14) What number of superficial yards, &c., are equivalent to 40253798 superficial inches?

Answer: 31060yds. 38in.

(15) Find the number of cubic yards, &c., in 141721 cubic inches; and prove it.

Answer: 3yds. 1ft. 25in.

(16) In 5279 pints, how many gallons, &c.? and prove the result.

Answer: 659gals. 3qts. 1pt.

(17) Required the number of weeks, &c., in 72015 hours; and verify the result.

Answer: 428wks. 4days. 15hrs.

(18) In 2706359 seconds, how many weeks, &c.? and prove it.

Answer: 4wks. 3days. 7hrs. 45min. 59sec.

(19) How many degrees, &c., are of equal value with 206265 seconds? and prove the converse.

Answer: $57^{\circ}.\ 17'.\ 45''$.

(20) In 12lbs. 10oz. 15dwts. 14grs. of silver, how many grains? and prove the result.

Answer: 74294 grains.

(21) Find how many grains there are in 18lbs. 2oz. 4drs. 2scr. 12grs.; and give a proof.

Answer: 104932 grains.

(22) In 20yds. 3qrs. 1nl., find the number of nails; and prove it.

Answer: 333 nails.

(23) What number of acres, &c., are equal in extent to 82973 square poles?

Answer: 518ac. 2ro. 13po.

(24) How many pints are equivalent to 987bar. 25gals. 3qts. 1pt. of ale? and prove it.

Answer: 284463 pints.

(25) Reduce 21tuns. 3hhds. 54gals. 2qts. of wine to pints; and the contrary.

Answer: 44284 pints.

(26) Required the number of quarts in 356qrs. 7bu. 2pks. 1gal. of corn; and prove it.

Answer: 91380 quarts.

(27) In 340 pistoles at 17*s.* 6*d.* each, how many pounds sterling?

Answer: £297. 10*s.*

(28) How many moidores of 27*s.* each, are equal to 198 guineas?

Answer: 154 moidores.

(29) In £453. 16*s.* 8*d.*, how many pieces of coin valued at 3*s.* 4*d.* each?

Answer: 2723 pieces.

(30) Find how often a rod of 2ft. 10in. in length, must be applied to measure 10 miles, 140 yards.

Answer: 18783 times, and 18in. over.

(31) What number of weights of 14oz. 13drs. each, are equivalent to 25cwt. 2qrs. 14lbs.?

Answer: 3100 weights, and 1oz. 4drs. over.

(32) How many revolutions will the wheel of a carriage, 4ft. 7in. in circumference, make in 2mi. 4fur.?

Answer: 2880 revolutions.

(33) If 5oz. of silk can be spun into a thread 2fur. 20po. long; what weight of silk would supply a thread sufficient to reach to the moon, if the distance be 240000 miles?

Answer: 107tons. 2cwt. 3qrs. 12lbs.

(34) A year being equivalent to 365 days 6 hours, find the number of years, &c., in 295402374 seconds.

Answer: 9yrs. 131days. 18hrs. 12min. 54sec.

59. Keeping in mind what was said in the first Article in this chapter, we need no additional inquiry to inform us that the fundamental operations on *Compound Quantities* must be performed as in *Integers*, with this difference, that instead of carrying and borrowing *tens*, we must do the same with the *different numbers* which connect their parts together: and we shall therefore merely enunciate the rule for each, at the beginning of the portion of the work appropriated to it.

I. COMPOUND ADDITION.

60. Rule. Arrange the quantities under one another according to their denominations: add together those of the lowest denomination: and having found the number of the next denomination to which the sum is equivalent, put down the remainder, if any, and add this number to those of the next denomination; and repeat the process till all the quantities are disposed of.

Ex. Find the sum of 142cwt. 1qr. 21lbs., 78cwt. 0qr. 14lbs., 21cwt. 2qrs. 19lbs., and 176cwt. 1qr. 15lbs.

The *form* of the operation is as underneath:

Common Operation.

cwt.	qrs.	lbs.	
142	1	21	
78	0	14	
21	2	19	
176	1	15	
cwt. 418	2	13	the sum.

Reductions.

lbs. lbs. qrs.
28) 69 (2
56

13lbs.:

qrs. qrs. cwt.
4) 6 (1
4

2qrs.

and the proof is the same as that of Article (23).

Examples for Practice.

(1) Add together £73. $2s.$ $9\frac{1}{2}d.$; £25. $8s.$ $4\frac{3}{4}d.$; £68. $3s.$ $11\frac{1}{4}d.$; £76. $17s.$ $7d.$, and £5. $14s.$ $5\frac{3}{4}d.$: and prove the result.

(2) Find the sum of 32cwt. 2qrs. 15lbs. 12oz.; 47cwt. 25lbs. 7oz.; 5cwt. 3qrs. 17lbs. 10oz.; 23cwt. 1qr. 19lbs. 15oz.; 1cwt. 2qrs. 10lbs. 8oz., and 9cwt. 3qrs. 14oz.: and prove it.

(3) Required the sum of 11yds. 2ft. 9in.; 46yds. 1ft. 8in.; 15yds. 1ft. 10in.; 38yds. 2ft. 9in.; 55yds. 11in., and 27yds. 2ft. 7in.: and prove it.

(4) Collect into one quantity, 49gals. 3qts. 1pt.; 34gals. 1qt.; 25gals. 1pt; 51gals. 3qts. 1pt.; 30gals. 1qt., and 53gals. 2qts. 1pt.: and prove it.

(5) Determine the aggregate of 10wks. 5days. 14hrs. 31min.; 18wks. 4days. 12hrs. 38min.; 25wks. 10hrs. 14min.; 75wks. 6days. 23hrs. 59min.; 53wks. 4days. 19hrs. 23min., and 40wks. 17hrs. 25min.: and prove it.

(6) Add together 64lbs. 11oz. 16dwts. 14grs.; 21lbs. 10oz. 12dwts. 13grs.; 2lbs. 1dwt. 16grs.; 12lbs. 10oz. 18grs.; 24lbs. 11oz. 12dwts., and 14lbs. 1oz. 1gr.: and prove the result.

(7) Find the sum of 11oz. 4drs. 2scrs. 11grs.; 10oz. 3drs. 4grs.; 11oz. 1scr. 14grs.; 10oz. 1scr. 16grs.; 2drs 2scrs. 18grs., and 14oz. 5drs. 1scr.: and prove it.

(8) Express in one sum, 21yds. 2qrs. 3nls.; 18yds. 2qrs. 2nls.; 21yds. 1qr. 2nls.; 16yds. 3qrs. 2nls; 12yds. 1qr. 2nls., and 14yds. 2qrs. 3nls.

(9) Find the sum of 32lea. 2mi. 1fur. 21po.; 16lea. 1mi. 3fur. 26po.; 18lea. 2mi. 6fur. 21po.; 13lea. 1mi. 2fur. 12po., and 26lea. 1mi. 4fur. 9po.

(10) Find the sum of 21ac. 1ro. 34po.; 16ac. 2ro. 27po.; 214ac. 1ro. 2po.; 32ac. 1ro. 28po., and 301ac. 14po.

II. COMPOUND SUBTRACTION.

61. Rule. Having properly arranged the quantities under one another, begin at the right hand and take each number in the lower line from the corresponding one in the upper, borrowing *instead* of 10, when necessary, the numbers which connect the successive denominations; and the several quantities thus obtained will be the remainder or difference.

Ex. Subtract 35yds. 2ft. 8in., from 48yds. 1ft. 4in.

Common Operation.

	yds.	ft.	in.	
	48	1	4	
	35	2	8	
yds.	12	1	8	the rem^r.

Reductions.

in. in. in.
16 = 4 + 12 borrowed,
8

8 in.

ft. ft. ft.
4 = 1 + 3 borrowed,
3 = 2 + 1 carried,

1 ft.

and the proof used for integers is applicable here.

Examples for Practice.

(1) Find the difference of £325. 19*s*. 4¾*d*. and £253. 18*s*. 6½*d*.: and prove it.

(2) Required the excess of 59tons. 13cwt. 2qrs. 23lbs. 11oz. 10drs. above 27tons. 17cwt. 1qr. 25lbs. 2oz. 14drs.; and verify the result.

(3) Subtract 82lea. 2mi. 5fur. 38po. from 281lea. 1mi. 7fur. 26po.; and verify it.

(4) Find the difference of 140gals. 3qts. 1pt. and 240gals.; and prove it.

(5) From 24days. 14hrs. 46min. 31sec., take 4 days. 21hrs. 18min. 52secs.; and verify it.

(6) Take 14lbs. 11oz. 12dwts. 19grs. from 81lbs. 10oz. 9dwts. 18grs.

(7) Required the difference of 28lbs. 7oz. 1dr. 2scr. 4grs. and 12lbs. 8oz. 2drs. 1scr. 12grs.

(8) What is the difference between 38ac. 31po. and 21ac. 3ro. 34po.?

(9) Subtract 1tun. 3hhds. 32gals. 4pts. of wine from 2tuns. 2hhds.

(10) Required the difference of 162qrs. 1bush. 1pk. and 127qrs. 4bush. 3pks. 1gal.

III. COMPOUND MULTIPLICATION.

62. Rule. Place the multiplier under the lowest denomination of the multiplicand, and find the number of the *next* denomination contained in the first product: put down the remainder, if any, and carry the quotient to the second product, and repeat the process till all the denominations are multiplied.

Ex. Multiply 35gals. 3qts. 1pt. by 7.

Common Operation.	*Reductions.*
gals. qts. pt.	pts. pts. qts. qts.
35 . 3 . 1	2) 7 4) 24
7	
gals. 251 . 0 . 1 the prodt.	qts. 3 . 1 pt: gals. 6 . 0 qt.

and this may be *proved* by reducing 35gals. 3qts. 1pt. to pints, multiplying the result by 7, and then reducing the product to gallons, &c.

When the multiplier exceeds 12, this process will be laborious, and we may use the following.

Ex. To find the product of 3days. 18hrs. 45min. by 47, we may have either of the operations below:

	days. hrs. min.			days. hrs. min.	
	3 . 18 . 45			3 . 18 . 45	
		$9 \times 5 + 2 = 47$			$8 \times 6 - 1 = 47$
	34 . 0 . 45			30 . 6 . 0	
	5			6	
	170 . 3 . 45			181 . 12 . 0	
	7 . 13 . 30			3 . 18 . 45	
da.	177 . 17 . 15	the prod^{t}.	da.	177 . 17 . 15	the prod^{t}.

Should this method require *many* factors to make up the multiplier, it would be better to reduce the multiplicand to the lowest denomination contained in it, to multiply this result by the multiplier, and then to reduce the product back again.

Examples for Practice.

(1) Multiply £358. 4*s*. $7\frac{3}{4}$*d*. by 5 and 9.

(2) Required the products of 49cwt. 3qrs. 15lbs. by 7 and 11.

(3) Find the products of 154yds. 2ft. 10in. by 6 and 10.

(4) Multiply 58gals. 3qts. 1pt. by 8 and 12.

(5) Multiply 42wks. 5days. 23hrs. 42min. by 3 and 4.

(6) Multiply £125. 15*s*. $9\frac{1}{4}$*d*. by 28 and 45.
Answers: £3522. 1*s*. 7*d*., and £5660. 9*s*. $8\frac{1}{4}$*d*.

(7) Multiply £53. 18*s*. $7\frac{3}{4}$*d*. by 51 and 83.
Answers: £2750. 10*s*. $11\frac{1}{4}$*d*., and £4476. 7*s*. $7\frac{1}{4}$*d*.

(8) Multiply 17cwt. 2qrs. 19lbs. 5oz. by 36 and 73.
Answers: 636cwt. 23lbs. 4oz., and 1290cwt. 9lbs. 13oz.

(9) Multiply 13lea. 2mi. 6fur. 25po. by 42 and 97.

Answers: 585lea. 1mi. 6fur. 10po., and 1352lea. 1mi. 2fur. 25po.

(10) Multiply 15qrs. 6bush. 3pks. 1gal. by 54 and 111.

Answers: 856qrs. 3bush. 1pk., and 1760qrs. 3bush. 1gal.

(11) Multiply 43days. 18hrs. 45min. by 77 and 147.

Answers: 3371days. 3hrs. 45min., and 6435days. 20hrs. 15min.

(12) Multiply $57^\circ . 7' . 45''$ by 4 and 6.

Answers: $228^\circ . 31'$., and $342^\circ . 46' . 30''$.

(13) What is the value of 72 reams of paper, at 13*s.* 8*d.* a ream?

Answer: £49. 4*s.*

(14) Find the cost of 120 ounces of silver, at 5*s.* $3\frac{3}{4}$*d.* an ounce?

Answer: £31. 17*s.* 6*d.*

(15) Find the number of yards in 40 pieces of cloth, each containing 42yds. 2qrs. 2nls.

Answer: 1705 yards.

(16) Required the price of 279cwt. at £3. 7*s.* $10\frac{1}{2}$*d.* a cwt.

Answer: £946. 17*s.* $1\frac{1}{2}$*d.*

(17) If I spend £2. 7*s.* $1\frac{1}{2}$*d.* a day, how much is that in a year of 365 days?

Answer: £860. 0*s.* $7\frac{1}{2}$*d.*

(18) What sum will purchase an estate of 2120 acres, when the price of each acre is £32. 5*s.* 6*d.*?

Answer: £68423.

(19) If each of 114 persons receive £1. 18*s.* $6\frac{1}{2}$*d.*, what is received by them all?

Answer: £219. 13*s.* 9*d.*

(20) How many pounds of silver are there in a half-dozen of dishes, each weighing 51oz. 10dwts., and a dozen of plates each weighing 15oz. 15dwts. 22grs.?

Answer: 41lbs. 6oz. 11dwts.

(21) If a wheel of 5yds. 1ft. 6in. in circumference make 64640 revolutions, what space will it pass over?

Answer: 202 miles.

IV. COMPOUND DIVISION.

63. Rule. Having placed the divisor and dividend as in integers, find how often the divisor is contained in the highest denomination of the dividend, put down the quotient; and reduce the remainder, if any, to the next inferior denomination, adding to it the number of that denomination in the dividend, and repeat the division: so proceed through all the denominations.

Ex. Divide 41wks. 6days. 19hrs. by 11.

Common Operation.

$$\begin{array}{r|l} & \text{wks. days. hrs.} \\ 11 & 41 \;.\; 6 \;.\; 19 \\ \hline & \text{wks. } 3 \;.\; 5 \;.\; 17 \text{ the quot}^{t}. \end{array}$$

Reductions.

$$\begin{array}{r|l} & \text{wks.} \\ 11 & 41 \\ \hline & \text{wks. } 3 \;.\; 8 \text{ weeks over:} \end{array}$$

$$\begin{array}{r|l} & \text{days.} \\ 11 & 62 = 8 \times 7 + 6, \\ \hline & \text{days } 5 \;.\; 7 \text{ days over:} \end{array}$$

$$\begin{array}{r|l} & \text{hrs.} \\ 11 & 187 = 7 \times 24 + 19, \\ \hline & \text{hrs. } 17. \end{array}$$

and the operation may be proved by that of multiplication.

When the divisor is greater than 12, the process may be conducted as in Article (42), if it be a composite number, and by long division, if it be incomposite.

Ex. To divide £1478. 13*s*. $8\frac{3}{4}$*d*. into 77 equal portions, we may use either of the subjoined methods:

£. s. d.

$$77\begin{cases} 7)\ 1478\ .\ 13\ .\ 8\frac{3}{4} \\ 11)\ \ 211\ .\ \ 4\ .\ 9\frac{3}{4}\ .\ 2 \end{cases}$$

£19. 4 . $0\frac{3}{4}$. 44f. over.

£. s. d. £. s. d.

77) 1478 . 13 . $8\frac{3}{4}$ (19 . 4 . $0\frac{3}{4}$
77
708
693
15
20
313
308
5
12
68
4
275
231
44f. over.

The division may also be effected by reductions analogous to those alluded to in Multiplication.

Examples for Practice.

(1) Divide £189. 8*s.* 4*d.* by 5 and 8.

(2) Find the quotients of 182cwt. 3qrs. 7lbs. by 7 and 9.

(3) Divide 1658yds. 1ft. by 6 and 10.

(4) Find the quotients of 238ac. 2ro. 32po. by 8 and 11.

(5) Divide 13wks. 5days. 19hrs. 30min. by 3 and 4.

(6) Divide 739qrs. 4bush. 2pks. 1gal. into 11 equal portions.

(7) What is the twelfth part of 22wks. 4 days. 20hrs. 43min. 24sec.?

(8) Divide £1738. 12*s.* $7\frac{1}{2}$*d.* by 18; and £1279. 13*s* $8\frac{3}{4}$*d.* by 23.

Answers: £96. 11*s.* $9\frac{3}{4}$*d.*, and £55. 12*s.* $9\frac{1}{4}$*d.*

(9) Divide 425tons. 15cwt. 2qrs. 12lbs. by 27; and 2374cwt. 1qr. 12lbs. 12oz. by 38.

Answers: 15tons. 15cwt. 1qr. 16lbs., and 62cwt. 1qr. 26lbs. 2oz.

(10) Find the quotient of 1361mi. 4fur. 28po. by 28; and of 3179lea. 1mi. 5fur. 16po. by 46.

Answers: 48mi. 5fur. 1po., and 69lea. 2fur. 36po.

(11) If 41cwt. cost £52. 10*s.* 7½*d.*, what is the price of a cwt.?

Answer: £1. 5*s.* 7½*d.*

(12) What will be the price of 1lb., when 1cwt. costs £137. 18*s.*?

Answer: £1. 4*s.* 7½*d.*

(13) If a soldier's pay for a year of 365 days be £9. 2*s.* 6*d.*; how much is that for a day?

Answer: 6*d.*

(14) If a person's yearly income be £65. 12*s.* 6*d.*, and he lay by £20. a year; how much does he spend each day?

Answer: 2*s.* 6*d.*

(15) If 145 sheep cost £169. 3*s.* 4*d.*; what is the price of a score at the same rate?

Answer: £23. 6*s.* 8*d.*

(16) A wheel makes 514 revolutions in passing over 1mi. 467yds. 1ft.? what is its circumference?

Answer: 4yds. 1ft.

(17) If a person complete a journey of 422mi. 3fur. 38po. in 37days; what distance does he travel each day?

Answer: 11mi. 3fur. 14po.

(18) If 8 packages of cloth, each consisting of 4 parcels, each parcel of 10 pieces, and each piece of 26 yards, cost £6656.; what is the price of a yard?

Answer: 16 shillings.

(19) If the clothing of 754 soldiers come to £3178. 11*s.* 7½*d.*; how much is that for each man?

Answer: £4. 4*s.* 3¾*d.*

(20) A vintner bought 138gals. of wine at 10*s*. a gallon, of which he retained 18gals. for his own use: at what rate per gallon must he sell the remainder, that he may have his own for nothing?

Answer: 11*s*. 6*d*.

(21) A ship's crew of 50 men have a supply of water for 30 days at 2 quarts a head: if they lose 125 gallons, and find that they will be 50 days at sea, what must be each man's daily allowance?

Answer: 1 quart.

64. The multipliers and divisors in the last two rules have been regarded as abstract numbers: and though it is impossible to determine the *product* of two concrete quantities *as such*, the *quotient* of one concrete magnitude by another of the same kind will be an *abstract* number, being merely the *number of times* one of them must be repeated to make up the other. See the *Appendix*.

Ex. The sum £263. 8*s*. 11½*d*. is distributed equally among a number of persons, so that the share of each is £37. 12*s*. 8½*d*: find the number of persons.

Here, the dividend = 252910 farthings:

and the divisor = 36130 farthings:

whence, the quotient is found to be 7, by common division: or, £37. 12*s*. 8½*d*. being *repeated* 7 times, amounts to £263. 8*s*. 11½*d*., and the number of persons is 7.

Hence, one concrete magnitude may be a measure or a multiple of another of the *same kind*.

CHAPTER III.

THE RULE OF THREE,

SOMETIMES CALLED THE GOLDEN RULE.

65. DEF. THE object of the *Rule of Three* is, by means of *three* quantities given, to determine a *fourth*, which shall be the same multiple, part or parts of one of them, that one of the remaining quantities is of the other; and it therefore follows that the operation, by which this may be accomplished, will depend upon those of Multiplication and Division already considered.

Ex. 1. If 1lb. of any commodity cost 3*s.* $4\frac{1}{2}$*d.*, it is required to find the price of 12lbs.

Here, it is evident that the required price will be the same multiple of 3*s.* $4\frac{1}{2}$*d.*, that 12lb. is of 1lb., which may therefore be found by *Multiplication:* thus,

$$\begin{array}{rl} \textit{s.}\ \ \textit{d.} & \\ 3\ .\ 4\tfrac{1}{2} & = \text{the price of 1lb.} \\ 12 & \\ \hline \pounds 2\ .\ 0\ .\ 6 & = \text{the price of 12lbs.:} \end{array}$$

and this result may be obtained by means of a *Statement* and *Operation* in the following *form*:

$$\begin{array}{ccccr} \text{lb.} & & \text{lbs.} & & \textit{s.}\ \ \textit{d.} \\ 1 & : & 12 & :: & 3\ .\ 4\tfrac{1}{2} \\ & & & & 12 \\ \hline & & & & 40 \\ & & & & 4 \\ \hline & & & & 162 \\ & & & & 12 \\ \hline & & & 4) & 1944\textit{f.} \\ \hline & & & 12) & 486\textit{d.} \\ \hline & & & 2,0) & 4,0\ .\ 6\textit{d.} \\ \hline & & & & \pounds 2\ .\ 0\ .\ 6, \text{ as before.} \end{array}$$

Ex. 2. If 11 bushels of wheat cost £4. 2*s*. 11½*d*., what sum must be paid for 45 bushels?

In this instance, we have, by *Division*,

```
         £. s. d.
     11 ) 4 . 2 . 11½
     ----------------
        £0 . 7 .  6½ = the price of 1 bushel:
```

and the required price will then be obtained by *Multiplication* as below:

```
       £.  s.  d.
       0 . 7 . 6½
               9 × 5 = 45
     ------------
       3 . 7 . 10½
               5
     ------------
    £16 . 19 . 4½ = the price of 45 bushels:
```

but we shall arrive at the same conclusion by conducting the solution of the question in a form similar to that of the last example: thus,

```
  bush.   bush.      £.  s.  d.
   11  :   45   ::   4 . 2 . 11½
                     20
                     --
                     82
                     12
                    ---
                    995
                      4
                   ----
                   3982
                     45
                  -----
                  19910
                 15928
                 ------
             11 ) 179190
                 ------
               4 ) 16290 f.
                 ------
              12 ) 4072½
                 ------
              2,0 ) 33,9 . 4
                 ----------
                £16 . 19 . 4½:
```

and it is easily seen that this sum is the same multiple and part of £4. 2*s*. 11½*d*., as 45 is of 11.

66. Proper attention to the processes here employed will enable us to embody their substance in the following general Rule.

RULE OF THREE.

For the Statement. Of the three quantities proposed, put down as the last on the right hand, that which is of the *same kind*, or under the *same circumstances* as the one required; and the *greater* or *less* of the two others in the second place, according as the required one ought, from the nature of the case, to be *greater* or *less* than the last; and the remaining one in the first place.

For the Operation. Reduce, if necessary, the first and second *terms* to the *same* denomination, and the third to the *lowest* denomination contained in it: multiply together the second and third terms thus reduced, and the quotient arising from the division of the product by the first, will be the quantity required, expressed in the *denomination* to which the *last* term was reduced: and it may be had in other denominations by the proper divisions or multiplications.

It is sometimes necessary to consider what *preparation* may be required before the rule is applied: and when the statement is made, the first, and the second *or* third terms may be divided by any factor common to them, either *before* or *after* the reductions, without affecting the result, inasmuch as no alteration is produced from Multiplication and Division by the same number.

Examples.

(1) If 4cwt. 3qrs. 10lbs. cost £30. 9*s*. 9*d*., what is the price of 12lbs.?

Here, what is required being *money*, the *last* term of the statement will be £30. 9*s*. 9*d*., since it is of the *same kind:* and because the price of 12lbs. must manifestly be *less* than that of 4cwt. 3qrs. 10lbs., the *second* term must be 12lbs., and the *first* will be 4cwt. 3qrs. 10lbs.: that is, the statement and operation will be as follows.

```
cwt. qrs. lbs.     lbs.      £.  s. d.
  4 . 3 . 10   :   12   ::  30 . 9 . 9
  4                         20
 ---                       ----
 19                        609
 28                         12
----                       ----
162                       7317
 38                         12
----                     ------
542               542 ) 87804 ( 162
                        542
                        ----
                        3360
                        3252
                        ----
                         1084
                         1084
```

and the answer will therefore be 162 *pence*, because the last term has been reduced to *that* denomination; or the price required = 162*d.* = 13*s.* 6*d.*

(2) The rental of a parish being estimated at £3766. 5*s.*, what sum will be raised by a rate of 8*d.* in the pound?

In this case all the quantities concerned are of the *same kind*, but the answer required being *rate* will be under the *same circumstances* as 8*d.*; and the statement and operation will therefore be

```
 £.      £.   s.      d.
  1  :  3766 . 5  ::  8
 20          20
 --     -------
 20     75325
```

that is, 20 : 75325 :: 8;

or, 4 : 15065 :: 8,

by dividing the *first* and *second* terms by 5:

or, 1 : 15065 :: 2,

by dividing the *first* and *third* terms by 4:
and by these steps the number of figures used is much diminished; so that we have the operation,

$$\begin{array}{l} 1 : 15065 :: 2 \\ \qquad\qquad\quad 2 \\ \hline 12\,)\,30130\,d. \\ \quad 2,0\,)\,251,0 \,.\, 10d. \\ \qquad\quad £12\,5.\ 10s.\ 10d. \end{array}$$

These two instances following out the rules given for the *Statement* and *Operation*, are regarded as exemplifications of the Rule of Three *Direct*, because *less* requires *less*, and *more* requires *more*.

(3) If a person can walk 300 miles in 8 days of 7hrs. 30min. each; in how many days can he do the same, when 10hrs. of each are available for the purpose?

Since *days* are here *required*, and the number of days is necessarily *less* as the number of hours employed in each is *greater*, we shall have,

$$\begin{array}{ccccc} \text{hrs.} & & \text{hrs. min.} & & \text{days.} \\ 10 & : & 7.30 & :: & 8 \\ 60 & & 60 & & \\ \hline 600 & & 450 & & \end{array}$$

$$\begin{array}{rrcrcr} \text{that is,} & 600 & : & 450 & :: & 8; \\ \text{or,} & 60 & : & 45 & :: & 8, \\ \text{or,} & 20 & : & 15 & :: & 8, \\ \text{or,} & 4 & : & 3 & :: & 8, \\ \text{or,} & 1 & : & 3 & :: & 2, \\ & & & 2 & & \\ \hline & & & 6 & \text{days.} & \end{array}$$

The reasons of the steps above taken are obvious, being similar to those in the preceding example, which are given at length, and the answer is 6 days: the *distance* 300 miles has not been taken into the consideration, because it is *common* to both conditions, and implies nothing more than *the same* or *an equal* distance or journey, whatever it might be, as may easily be made to appear; for, 8 days of 7hrs. 30min. each, give 60hrs. for completing the journey of 300 miles, so that he walks 5 miles an hour: and in 6 days of 10hrs. each, he would finish the same at the *same rate:* and this rate the 300 miles has enabled us to find, but *it* has nothing to do with the *question*.

(4) What length of carpet 2ft. 3in. wide, will be required to cover a room which is 27ft. 6in. long, and 22ft. 6in. wide?

The *length* of the piece of carpet being required, we shall have the statement and operation as follows:

$$\begin{array}{lcccccr}
 & \text{ft. in.} & & \text{ft. in.} & & \text{ft. in.} & \\
 & 2\,.\,3 & : & 22\,.\,6 & :: & 27\,.\,6 & \\
 & 12 & & 12 & & 12 & \\
 & 27 & : & 270 & :: & 330 & \\
\text{or,} & 1 & : & 10 & :: & 330 & \\
 & & & & & 10 & \\
 & & & 12\,)\,3300 \text{ in.} & & & \\
 & & & & 3\,)\,275 \text{ ft.} & & \\
 & & & & & 91 \text{ yds. } 2\text{ft.} &
\end{array}$$

These two examples may be said to belong to the Rule of Three *Inverse*, from the circumstance that *more* requires *less*, and *less* requires *more:* but as the rules for the *Statement* and *Operation* above given are *universal*, it does not seem desirable in this part of Arithmetic to make any distinction between the *Rule of Three Inverse*, and the *Rule of Three Direct.*

(5) A person gives away annually £20. in charity, and his weekly bills amount to £7. 10*s.*: what additional daily expenditure may he incur with an income of £592. 10*s.*?

For the annual amount of his weekly bills, we have

$$\begin{array}{lcccccr}
 & \textit{wk.} & & \textit{wks.} & & \text{£.}\ \ \textit{s.} \\
 & 1 & : & 52 & :: & 7\,.\,10 \\
\text{or,} & 1 & : & 52 & :: & 150\,\textit{s.} \\
 & & & & & 52 \\
 & & & & & 300 \\
 & & & & & 750 \\
 & & & & 2{,}0\,) & 780{,}0\,\textit{s.} \\
 & & & & & \text{£}\,390.
\end{array}$$

therefore, his charity and weekly bills amount to £20. together with £390., which is £410.: and he has the excess of £592. 10*s.* above £410. which is £182. 10*s.* left to be expended in 365 days:

$$\begin{array}{lcccccl} & \textit{days.} & & \textit{day.} & & \pounds. \quad s. & \\ \text{whence,} & 365 & : & 1 & :: & 182\,.\,10 & \\ & & & & & 20 & \\ \hline & & & & 365 \,) & 3650 & (\,10s. \\ & & & & & 3650 & \end{array}$$

and consequently he may spend 10*s.* a day.

Considerations of the *nature* of those here introduced, are generally *simple* in their character; and with a little attention, the rule we are discussing may be made available for conducting most of the *ordinary* affairs of mankind; as far, at least, as they depend upon the connection implied by the expression, *Cause and Effect.*

Examples for Practice.

(1) Required the price of 450lbs., at 4*s.* 8½*d.* a lb.

Answer: £105. 18*s.* 9*d.*

(2) Find the amount of a servant's wages for 215 days, at 2*s.* 4¼*d.* a day.

Answer: £25. 6*s.* 1¾*d.*

(3) A person's salary is £191. 12*s.* 6*d.* for 365 days: in how many days will he have a claim for £31. 10*s.*?

Answer: 60 days.

(4) If 25cwt. 2qrs. cost £7. 6*s.* 7½*d.*, how much is that for 1cwt.?

Answer: 5*s.* 9*d.*

(5) Required the price of 4cwt. 1qr. 4lbs. 8oz., when 1lb. costs 7*s.* 10½*d.*

Answer: £189. 3*s.* 11¼*d.*

(6) If 6yds. 3qrs. cost 5*s.* 3*d.*, how much will 73yds. 2qrs. cost, at the same rate.?

Answer: £2. 17*s.* 2*d.*

(7) If an artificer earn £19. 1*s.* in 20 days; in what time will he earn £23. 16*s.* 3*d.*?

Answer: 25 days.

(8) If a person walk 216 miles in 7 days of 16 hours each; in how many days of 12 hours each can he do the same?

Answer: 9 days, 4 hours.

(9) If 90 English degrees be equivalent to 100 French degrees, how many English degrees, &c. will be contained in 654 French degrees?

Answer: 588deg. 36min.

(10) If 17ells. 3qrs., each ell containing 5qrs., be bought for £6. 17*s.* 6*d.*: how much must be paid for 18yds.?

Answer: £5. 12*s.* 6*d.*

(11) If 1000 sovereigns weigh 21 lbs. 5 oz. 16 dwts. 6 grs., what weight of gold will be contained in 384 sovereigns?

Answer: 8 lbs. 3 oz.

(12) How much wheat can be purchased for £55. 0*s.* 3*d.*, at the rate of 6*s.* 9½*d.* a bushel?

Answer: 20qrs. 2bush.

(13) If a farm of 375 acres, be let for £401. 11*s.* 3*d.* a year, what is that for each acre?

Answer: £1. 1*s.* 5*d.*

(14) If lodgings be let at 13*s.* 6*d.* a week, what will the demand amount to for 273 days?

Answer: £26. 6*s.* 6*d.*

(15) Required the price of 36cwt. 1qr., when 2cwt. 2qrs. 10lbs. cost £4. 7*s.* 9½*d.*?

Answer: £61. 9*s.* 1*d.*

(16) If a servant's wages be £30. 0*s.* 8¾*d.* a year, what will be his demand for a service of 338 days?

Answer: £27. 16*s.* 3½*d.*

(17) If a person can walk 3mi. 6fur. 25po. in an hour, in what time will he complete a journey of 99mi. 4fur. 10po.?

Answer: 26 hours.

(18) What is the cost of 19bar. 24gals. 3qts. 1pt. of beer, at 3½*d.* a quart?

Answer: £41. 7*s.* 0¼*d.*

(19) If the carriage of 3cwt. 2qrs. 14lbs. for 51

miles come to 18*s.* $5\frac{1}{4}$*d.*; what will be the charge for carrying 10tons. 3cwt. the same distance?

Answer: £51. 12*s.* 6*d.*

(20) A bankrupt owes £3840., and his whole property amounts to no more than £828.: what dividend will his creditors receive in the pound?

Answer: 4*s.* $3\frac{3}{4}$*d.*

(21) At the rate of 11*s.* $7\frac{1}{2}$*d.* in the pound, what is the sum paid by a bankrupt for a debt of £2735. 10*s.*?

Answer: £1590. 0*s.* $2\frac{1}{4}$*d.*

(22) If a labourer earn 2*s.* a day when wheat is at 8*s.* a bushel, what ought he to earn when wheat is at 6*s.* a bushel?

Answer: 1*s.* 6*d.*

(23) If a tradesman gain 1*s.* $4\frac{1}{2}$*d.* on an article which he sells for 5*s.* 6*d.*, what does he gain on every £100.?

Answer: £25.

(24) If 15 workmen can do a piece of work in 25 days, in what time can 25 men do the same?

Answer: 15 days.

(25) How much in length, that is 3ft. 9in broad, will be equivalent to 37ft. 9in. in length, which is 7ft. 6in. broad?

Answer: 75ft. 6in.

(26) If 69yds. of carpet 3qrs. wide, cover a room 8yds. 2qrs. 2nls. long; find the width of the room.

Answer: 6 yards.

(27) If by paying down £89. 2*s.* 6*d.*, I become entitled to £3. a year, what income shall I derive from disposing of £1002. 13*s.* $1\frac{1}{2}$*d.*, in the same way?

Answer: £33. 15*s.*

(28) What will be the purchase-money of an estate producing a rental of £3223., at the rate of £2. 15*s.* for every £100.?

Answer: £117200.

(29) If a person's annual income be 650 guineas, how much will he have saved at the end of the year, after spending £10. 13*s.* $9\frac{1}{2}$*d.*, a week?

Answer: £126. 12*s.* 10*d.*

(30) What may a person having an income of £1000. a year, spend daily, so as to lay by £434. 5*s.* yearly?

Answer: £1. 11*s.*

(31) If I lend a friend £250. for 6 months, how long ought he to lend me £187. 10*s.*, to requite the kindness?

Answer: 8 months.

(32) What is the tax upon £302. 3*s.* 7*d.*, when £429. 8*s.* 3*d.* is rated at 13*s.* 6*d.*?

Answer: 9*s.* 6*d.*

(33) If the rate levied upon a rental of £763. 15*s.* amount to £133. 13*s.* 1½*d.*, how much is it in the pound?

Answer: 3*s.* 6*d.*

(34) A person buys 136yds. of cloth for £150., and retails it at £1. 18*s.* a yard; what does he gain by the transaction?

Answer: £108. 8*s.*

(35) A person's daily income is £1. 15*s.*, and his quarterly expenditure is £135. 10*s.*; how much will he have saved at the end of 9 years?

Answer: £870. 15*s.*

(36) If a gentleman spend £152. 10*s.* every week; what must be his daily income that in 15 years he may lay by £7522. 10*s.*?

Answer: £23. 2*s.*

(37) A person bought 180 gallons of wine for £125.: find the quantity of water to be added that he may retail the mixture at 12*s.* 6*d.* a gallon.

Answer: 20 gallons.

(38) If an estate produce £1680. a year, and the land-tax be payable upon this sum at 3*s.* 6*d.* in the pound: what is its clear annual value?

Answer: £1386.

(39) When a bankrupt's effects pay three dividends of 4*s.* 2¼*d.*, 3*s.* 2¼*d.*, and 2*s.* 4½*d.* in the pound: what do his creditors lose upon his entire debt, which is £4265?

Answer: £2185. 16*s.* 3*d.*

(40) A person bought 125 yards of cloth, at the rate of 2 yards for 5*s.*, and 125 yards at the rate of 3 yards for 5*s.*: what will he gain or lose by selling the 250 yards at the rate of 5 yards for 10*s.*?

Answer: He will lose £1. 0*s.* 10*d.*

67. Questions frequently occur, in which it is necessary to repeat the process just explained, and they are on this account said to belong to the *Double Rule of Three*: but we shall here adapt what has already been done, to the solution of a single example, which will be sufficient to point out the steps to be pursued in every other instance.

Ex. If a person travel 300 miles in 10 days, when the day is 12 hours long; how many days will it take him to travel 600 miles, when the day is 15 hours long?

We here give two solutions, each of which depends upon the last Article.

First Solution.

mi. mi. days.
300 : 600 :: 10
10
3,00)60,00
20 days, in which he will travel 600 miles, when the days are 12 hours long:

hrs. hrs. days.
15 : 12 :: 20
20
15)240
16 days of 15 hours each, in which he will travel 600 miles.

Second Solution.

hrs. hrs. days.
15 : 12 :: 10
10
15)120
8 days of 15 hours each, in which he will travel 300 miles:

mi. mi. days.
300 : 600 :: 8
8
3,00)48,00
16 days of 15 hours each, in which he will travel 600 miles.

Examples for Practice.

(1) If the expenses of 7 persons for 3 months amount to 70 guineas; what will be the expenditure of 10 persons for 12 months at the same rate?

Answer: £420.

(2) If 10 horses consume 7bush. 2pks. of oats in 7 days; in what time will 28 horses consume 3qrs. 6bush. at the same rate?

Answer: 10 days.

(3) If 10 men reap 20 acres of corn in 4 days; how many men can reap 70 acres in 10 days, at the same rate of labour?

Answer: 14 men.

(4) If 48 men can do a piece of work in 16 days of 9 hours each: in how many days of 12 hours each will 64 men be able to do a piece of work three times as great?

Answer: 27 days.

(5) If the carriage of 13cwt. 2qrs. 19lbs. for 35 miles come to £4. 17*s*. 6*d*.; what must be paid for the conveyance of 41cwt. 1lb. for 49 miles?

Answer: £20. 9*s*. 6*d*.

(6) If £20. in trade gain £16. in 15 months, what sum will gain £24. in 3 months, at the same rate?

Answer: £150.

(7) If 12 men can perform a piece of work in 20 days; required the number of men who could perform another piece of work four times as great in a fifth part of the time.

Answer: 240 men.

(8) If with a capital of £1000., a tradesman gain £100. in 7 months, in what time will he gain £60. 10*s*., with a capital of £385.?

Answer: 11 months.

CHAPTER IV.

THE DOCTRINE OF FRACTIONS,

USUALLY TERMED VULGAR FRACTIONS.

68. Def. All whole numbers or *Integers*, being supposed to be formed by the *repetition* of the unit, may therefore be regarded as the result of the *multiplication* of that element; but if the unit be considered capable of *division* into any number of *equal* portions, the quantities thence arising must be viewed in the light of *broken* magnitudes; and these are therefore termed *Fractions*, or more generally, *Vulgar Fractions*, in order to distinguish them from fractions of a different *form*, whose nature will be discussed in the next chapter.

NOTATION AND NUMERATION OF FRACTIONS.

69. Def. 1. If we suppose the *unit* to be divided into 2, 3, 4, 5, &c., equal portions, *one* of the portions in each case is represented by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., which may be regarded as the *primitive Fractions* of their respective denominations, and are called the *Reciprocals* of the natural numbers, 2, 3, 4, 5, &c.: also, the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., are read, *one-half, one-third, one-fourth, one-fifth*, &c.

70. Def. 2. If *two* or *more* of these equal portions be taken together, the *aggregates* thence arising are expressed by repeating the unit as *often* as such portions are repeated, in the *form* of their sum, the number below the line remaining the same.

Thus, if the primitive fraction $\frac{1}{3}$ be taken *twice*, there will arise a new fraction expressed by $\frac{2}{3}$: if $\frac{1}{4}$ be repeated *thrice*, there results a new fraction expressed by $\frac{3}{4}$: again, if $\frac{1}{5}$ be taken *four times*, the new fraction will be $\frac{4}{5}$; and similarly of all the other primitive fractions: also, the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, &c., are read *two-thirds, three-fourths, four-fifths*, &c.: and all quantities of this *form* are called *Simple Fractions*.

71. DEF. 3. Hence, the number *below* the line denotes the number of equal portions into which the unit is supposed to be divided, and is therefore called the *Denominator;* and the number *above* the line expressing the number of such equal portions intended to be taken, is therefore termed the *Numerator.*

Thus, of the fraction $\frac{5}{7}$, whose *Terms* are 5 and 7, the denominator 7 implies that the unit is supposed to be divided into *seven* equal portions; and the numerator 5 shews that *five* of such equal portions are here the object of our consideration: and hence it is also manifest, that the *integer* 5 is 7 times as great as the *fraction* $\frac{5}{7}$; and 5 may therefore be expressed in a *Fractional Form* by $\frac{5}{1}$.

72. From the last Article it follows, that if the numerator be less than the denominator, the value of the fraction is less than the unit; if the numerator be equal to the denominator, the value of the fraction is the unit; and if the numerator be greater than the denominator, the value of the fraction is greater than the unit.

73. DEF. 4. If the numerator be less than the denominator, the fraction is termed a *Proper Fraction;* but if the numerator be greater than the denominator, it is called an *Improper Fraction:* also, if the terms be equal to one another, we have merely the representation of the unit in the *form* of a fraction.

Thus, $\frac{2}{5}$ is a proper fraction, $\frac{11}{6}$ is an improper fraction, and $\frac{7}{7}$ is a representation of the unit in a fractional form, being of the same value as $\frac{8}{8}$, $\frac{9}{9}$, &c.

74. We are hence enabled to find the results of the multiplication and division of a fraction by an integer, and these may be integers or fractions.

If the fraction $\frac{4}{13}$ be multiplied by 3, the product is evidently $\frac{4 \times 3}{13} = \frac{12}{13}$; because in $\frac{12}{13}$, *three times* as many parts of the unit are implied, as there are in $\frac{4}{13}$.

If the fraction $\frac{2}{7}$ be divided by 5, the quotient will be $\frac{2}{7 \times 5} = \frac{2}{35}$; because the same numbers of parts are

indicated in $\frac{2}{7}$ and $\frac{2}{35}$, and each part in the former is *five times* as great as each part in the latter, by Article (71).

Hence, to *multiply* and *divide* a fraction by a whole number, we have only to multiply the *numerator* and *denominator* by it, respectively.

75. What is called a *Compound Fraction*, may be replaced by a simple one, by similar reasoning.

A *Compound Fraction* is made up of two or more simple fractions connected by the word *of*, as $\frac{1}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$:

$$\left.\begin{aligned} \text{now, } \frac{1}{5} \text{ of } \frac{6}{7} &= \frac{6}{7} \div 5 = \frac{6}{35} \\ \text{and } \frac{4}{5} \text{ of } \frac{6}{7} &= \frac{6}{35} \times 4 = \frac{24}{35} \end{aligned}\right\} \text{by the last Article:}$$

whence, $\frac{1}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$ is evidently the same as

$$\frac{1}{3} \text{ of } \frac{24}{35} = \frac{24}{35} \div 3 = \frac{24}{105},$$

a *simple* fraction of the ordinary form: that is,

$$\frac{1}{3} \text{ of } \frac{4}{5} \text{ of } \frac{6}{7} = \frac{1 \times 4 \times 6}{3 \times 5 \times 7} = \frac{24}{105}:$$

and from this, we infer that a compound fraction is equivalent to the simple fraction formed by multiplying together respectively the numerators and the denominators of its constituent simple fractions.

TRANSFORMATION OF FRACTIONS.

76. *If the numerator and denominator of a fraction be multiplied or divided by the same number, the value of the fraction will not be altered.*

For, if the fraction $\frac{3}{7}$ be multiplied by 5, the product is $\frac{15}{7}$: and again if this be divided by 5, the quotient is $\frac{15}{35}$, by the last Article but one: but since these two operations are the *reverse* of, and therefore *neutralize*, each other, it follows that

$$\frac{3}{7} = \frac{15}{35} = \frac{3 \times 5}{7 \times 5};$$

and also, that

$$\frac{15}{35} = \frac{3}{7} = \frac{15 \div 5}{35 \div 5}.$$

Hence, a whole number may be expressed in the form of a fraction with *any* denominator we please: thus,

$$5 = \frac{5}{1} = \frac{10}{2} = \frac{15}{3} = \frac{20}{4} = \&c.$$

Also, a fraction may be transformed into another with a *given* denominator, provided it be a *multiple* of the denominator of the proposed fraction: thus, $\frac{7}{8}$ may be transformed so as to have 96 for a denominator, because

$$\frac{7}{8} = \frac{7 \times 12}{8 \times 12} = \frac{84}{96}.$$

77. Since

$$\frac{5}{8} \times 4 = \frac{20}{8} = \frac{5 \times 4}{2 \times 4} = \frac{5}{2};$$

for the *Multiplication* of a fraction by an integer, it appears to be immaterial whether the numerator be multiplied, or the denominator be divided, by it: and inasmuch as

$$\frac{8}{9} \div 4 = \frac{8}{36} = \frac{2 \times 4}{9 \times 4} = \frac{2}{9};$$

for the *Division* of a fraction by a whole number, it amounts to the same thing whether we divide the numerator, or multiply the denominator, by it.

78. *A quantity made up of two others, one of which is an integer and the other a fraction, may be represented in the form of a fraction alone.*

Let us take $3\frac{4}{5}$, which is called a *mixed* quantity, and is intended to express the integer 3 and the fraction $\frac{4}{5}$ taken together, and is read *three* and *four-fifths*: then, since

$$3 = \frac{3}{1} = \frac{3 \times 5}{1 \times 5} = \frac{15}{5},$$

the mixed quantity $3\frac{4}{5}$ is equivalent to $\frac{15}{5}$ and $\frac{4}{5}$ taken together, or, to $\frac{19}{5}$ by Article (70): and this operation put in the *form*,

$$3\tfrac{4}{5} = \frac{3 \times 5 + 4}{5} = \frac{15 + 4}{5} = \frac{19}{5}$$

gives the following Rule.

RULE. Multiply the integer by the denominator of the fraction: to the product add the numerator, and the result will be the new numerator, which placed over the denominator will form the *improper* fraction required.

Ex. Represent $121\frac{5}{11}$ and $344\frac{24}{29}$ as improper fractions.

$$121\tfrac{5}{11} = \frac{121 \times 11 + 5}{11} = \frac{1331 + 5}{11} = \frac{1336}{11}$$

$$\begin{array}{r} 344\frac{24}{29} \\ 29 \\ \hline 3096 \\ 688 \\ 24 \\ \hline 10000 \\ \hline 29 \end{array}$$

or, $\frac{1336}{11}$ and $\frac{10000}{29}$ are the fractions required; and the second operation is made to differ in *form* from the first, only because the denominator in the latter quantity is beyond the extent of the Multiplication Table.

Examples for Practice.

(1) Express the mixed quantities $2\frac{5}{7}$, $5\frac{4}{9}$, $12\frac{7}{13}$ and $54\frac{8}{11}$ as improper fractions.

(2) Put into fractional forms, the mixed quantities $41\frac{7}{13}$, $123\frac{4}{17}$, $275\frac{14}{15}$ and $374\frac{54}{103}$.

79. Hence, a compound fraction formed of mixed quantities, may by Article (75) be put in the form of a simple fraction.

Thus, $2\frac{2}{3}$ of $5\frac{1}{6} = \frac{8}{3}$ of $\frac{31}{6} = \frac{8 \times 31}{3 \times 6} = \frac{248}{18} = \frac{124}{9}$.

Examples for Practice.

Exhibit $\frac{5}{7}$ of $3\frac{1}{8}$; $4\frac{2}{5}$ of $\frac{8}{9}$; $\frac{3}{4}$ of $\frac{4}{9}$ of $12\frac{1}{3}$, and $15\frac{7}{11}$ of $8\frac{4}{5}$ of $13\frac{3}{7}$, as improper fractions.

80. By means of the preceding Articles, what is called a *Complex Fraction* may be transformed into a simple fraction.

$$\text{Thus, } \frac{2\frac{1}{5}}{3\frac{2}{9}} = \frac{\frac{11}{5}}{\frac{29}{9}} = \frac{\frac{11}{5} \times 5 \times 9}{\frac{29}{9} \times 9 \times 5} = \frac{11 \times 9}{29 \times 5} = \frac{99}{145},$$

a simple fraction, obtained by multiplying the numerator and the denominator of the complex fraction, when expressed as improper fractions, by the product of the denominators: and this is most easily effected by placing them in *different* orders, as above.

Ex. Simplify the expressions $\frac{5\frac{7}{9}}{9\frac{5}{11}}$ and $\frac{6\frac{4}{9}}{13\frac{1}{14}}$ of $\frac{8\frac{5}{7}}{23\frac{1}{5}}$.

$$\text{Here, } \frac{5\frac{7}{9}}{9\frac{5}{11}} = \frac{\frac{52}{9} \times 9 \times 11}{\frac{104}{11} \times 11 \times 9} = \frac{52 \times 11}{104 \times 9} = \frac{1 \times 11}{2 \times 9} = \frac{11}{18}:$$

$$\text{and } \frac{6\frac{4}{9}}{13\frac{1}{14}} \text{ of } \frac{8\frac{5}{7}}{23\frac{1}{5}} = \frac{58 \times 14}{183 \times 9} \text{ of } \frac{61 \times 5}{116 \times 7} = \frac{5}{27}.$$

The numerators and denominators should in the *progress* of the operations, be divided by such common factors as may be found by *inspection*.

Examples for Practice.

(1) What are the simple fractions equivalent to the complex fractions $\frac{4\frac{3}{4}}{5\frac{4}{5}}$, $\frac{8\frac{8}{11}}{14\frac{6}{7}}$, $\frac{9\frac{7}{9}}{12\frac{5}{6}}$ and $\frac{25\frac{7}{8}}{34\frac{11}{16}}$?

Answers: $\frac{95}{116}$, $\frac{84}{143}$, $\frac{16}{21}$ and $\frac{138}{185}$.

(2) Simplify $\frac{7\frac{5}{6}}{9\frac{7}{8}}$ of $\frac{14\frac{4}{11}}{13\frac{3}{7}}$, and $5\frac{1}{7}$ of $\frac{6\frac{3}{5}}{10\frac{5}{7}}$ of $\frac{15}{12\frac{4}{7}}$ of $\frac{8\frac{1}{3}}{42}$.

Answers: $\frac{28}{33}$ and $\frac{3}{4}$.

81. *A quantity in the form of an improper fraction, may be expressed by a mixed quantity.*

We see immediately that $\frac{35}{8}$ is equivalent to $\frac{32+3}{8}$, or, to $\frac{32}{8}$ and $\frac{3}{8}$ taken together: but $\frac{32}{8}$ is equal to the integer 4, and therefore the required mixed quantity will be equal to the integer 4 and the proper fraction

$\frac{3}{8}$ taken together, whioh is sometimes expressed by $4 + \dfrac{3}{8}$, but generally in the form $4\frac{3}{8}$.

This process is evidently the same thing as dividing the numerator and denominator by the *denominator*, and noticing the *remainder* of the former: thus, for the improper fractions $\dfrac{327}{11}$ and $\dfrac{9999}{31}$, we may proceed according to the *forms:*

$$\begin{array}{r} 11\,)\,327 \\ \hline 29\frac{8}{11} \end{array} \qquad\qquad \begin{array}{r} 31\,)\,9999\,(\,322\frac{17}{31} \\ 93\phantom{(322\frac{17}{31}} \\ \hline 69\phantom{(322\frac{17}{31}} \\ 62\phantom{(322\frac{17}{31}} \\ \hline 79\phantom{(322\frac{17}{31}} \\ 62\phantom{(322\frac{17}{31}} \\ \hline 17\phantom{(322\frac{17}{31}} \end{array}$$

which are both implied in the following Rule.

RULE. Divide the numerator by the denominator, and the quotient will be the integral part; and the fractional part will be formed by placing the remainder over the denominator. If there be no remainder, the fraction is equivalent to the integer thus found.

Examples for Practice.

(1) Find the mixed quantities equivalent to

$$\frac{7}{3},\ \frac{19}{5},\ \frac{38}{9},\ \frac{149}{11} \text{ and } \frac{199}{12}.$$

(2) Express $\dfrac{440}{13}$, $\dfrac{2417}{19}$, $\dfrac{3797}{29}$ and $\dfrac{30471}{37}$, as mixed quantities.

(3) Represent $\frac{1}{3}$ of $\frac{4}{7}$ of $6\frac{3}{5}$, and $\dfrac{5}{6}$ of $\dfrac{13\frac{2}{5}}{4\frac{7}{9}}$, in the forms of mixed quantities.

82. *A fraction may be reduced to its lowest terms, by dividing its numerator and denominator, by their greatest common measure.*

For, since the value of a fraction is not altered by dividing its terms by any factor common to them both,

it will necessarily be expressed in its *lowest* or *simplest* terms, when that factor is their *greatest* common measure.

If the greatest common measure be 1, the fraction is already in its lowest terms.

Ex. Express the fraction $\frac{825}{960}$ in its lowest terms.

By Article (52), we have

$$\begin{array}{rl}
825)\,960\,(1 & \\
825 & \\
\hline
135)\,825\,(6 & \\
810 & \\
\hline
15)\,135\,(9 & \\
135 & \\
\hline
\end{array}$$

so that 15 is the greatest common measure: whence, dividing the terms of the fraction by it, as follows:

$$\begin{array}{r|r}
15)\,825\,(55 & 15)\,960\,(64 \\
75 & 90 \\
\hline
75 & 60 \\
75 & 60 \\
\hline
\end{array}$$

we have $\frac{55}{64}$ for the *equivalent* fraction expressed in the least terms possible.

The terms of the *original* fraction are *equal* multiples, or *equimultiples*, of those of the *reduced* one.

83. In many instances it is unnecessary to find the greatest common measure at *first*, the fractions being reducible to lower terms by successive divisions of the numerators and denominators by common factors discovered by *inspection*.

$$\text{Thus, } \frac{4968}{5904}=\frac{2484}{2952}=\frac{1242}{1476}=\frac{621}{738}=\frac{207}{246}=\frac{69}{82},$$

from *three* successive divisions of the numerator and denominator by 2, and then from *two* successive divisions by 3: and these are the terms which would have been obtained from dividing at *once* by 72, which is their greatest common measure.

This step may be advantageously used, when the terms are large, to diminish the labour of finding the greatest common measure by the general method.

Examples for Practice.

(1) Find the values of $\frac{9}{24}$, $\frac{63}{144}$, $\frac{147}{189}$, $\frac{435}{957}$ and $\frac{555}{999}$ in their lowest terms.

Answers: $\frac{3}{8}$, $\frac{7}{16}$, $\frac{7}{9}$, $\frac{5}{11}$ and $\frac{5}{9}$.

(2) Express in their simplest forms, the fractions, $\frac{3094}{3042}$, $\frac{3444}{3556}$, $\frac{5239}{6076}$, $\frac{5565}{8533}$ and $\frac{7568}{9504}$.

Answers: $\frac{119}{117}$, $\frac{123}{127}$, $\frac{169}{196}$, $\frac{15}{23}$ and $\frac{43}{54}$.

(3) Find the simplest fractions expressive of the values of $\frac{13667}{14186}$, $\frac{13478}{16701}$, $\frac{8398}{29393}$, $\frac{43365}{44688}$ and $\frac{48510}{49005}$.

Answers: $\frac{79}{82}$, $\frac{46}{57}$, $\frac{2}{7}$, $\frac{295}{304}$ and $\frac{98}{99}$.

(4) Simplify as much as possible the fractions, $\frac{11050}{35581}$, $\frac{20301}{33633}$, $\frac{714285}{999999}$, $\frac{109375}{1000000}$ and $\frac{135795}{222210}$.

Answers: $\frac{50}{161}$, $\frac{67}{111}$, $\frac{5}{7}$, $\frac{7}{64}$ and $\frac{11}{18}$.

84. *Two or more fractions having different denominators, may be transformed into equivalent fractions having a common denominator.*

Let it be required to express $\frac{1}{2}$, $\frac{2}{5}$ and $\frac{3}{7}$ with a common denominator; then, since the continued product of the denominators is $2 \times 5 \times 7$, we have

$$\frac{1}{2} = \frac{1 \times 5 \times 7}{2 \times 5 \times 7} = \frac{35}{70}:$$

$$\frac{2}{5} = \frac{2 \times 2 \times 7}{5 \times 2 \times 7} = \frac{28}{70}:$$

$$\frac{3}{7} = \frac{3 \times 2 \times 5}{7 \times 2 \times 5} = \frac{30}{70}:$$

so that $\frac{35}{70}$, $\frac{28}{70}$ and $\frac{30}{70}$ are the new *equivalent* fractions with the common denominator 70; and the steps taken are comprised in the operations here subjoined:

$$\text{first, } \left.\begin{array}{r} 1 \times 5 \times 7 = 35 \\ 2 \times 2 \times 7 = 28 \\ 3 \times 2 \times 5 = 30 \end{array}\right\} \text{ the new numerators:}$$

and $2 \times 5 \times 7 = 70$, the common denominator:

wherefore the equivalent fractions are $\frac{35}{70}$, $\frac{28}{70}$ and $\frac{30}{70}$, as above: and hence we derive the following Rule.

RULE. Multiply each numerator by all the denominators *except* the one placed under it, for the new numerator: and multiply all the denominators together for the common denominator.

85. If two or more of the denominators have a common measure, the equivalent fractions may be expressed in simpler terms than obtainable by the Rule, and still having a common denominator: thus, if the fractions be $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$, we have from Article (54),

$$\frac{2 \times 3}{1} = 6, \text{ and } \frac{6 \times 4}{2} = 12,$$

the least common multiple of the denominators: then,

$$\left.\begin{array}{l} \dfrac{1}{2} = \dfrac{1 \times 6}{2 \times 6} = \dfrac{6}{12} \\[2ex] \dfrac{2}{3} = \dfrac{2 \times 4}{3 \times 4} = \dfrac{8}{12} \\[2ex] \dfrac{3}{4} = \dfrac{3 \times 3}{4 \times 3} = \dfrac{9}{12} \end{array}\right\} \text{are the equivalent fractions,}$$

with the *least* common denominator 12; and the new numerators are here obtained by multiplying those of the fractions proposed by the quotients arising from *its* division by their respective denominators.

Fractions may be *compared* by means of this rule: and we see that *mixed quantities, compound* and *complex*

fractions must be reduced to simple fractions, before it can be applied.

Examples for Practice.

(1) Bring $\frac{2}{5}$ and $\frac{4}{5}$; $\frac{1}{7}$ and $\frac{2}{9}$; $\frac{3}{4}$ and $\frac{9}{11}$, respectively, to common denominators.

Answers: $\frac{10}{15}, \frac{12}{15}; \frac{9}{63}, \frac{14}{63}; \frac{33}{44}, \frac{36}{44}.$

(2) Reduce to, or express with, a common denominator, $\frac{1}{2}$, $\frac{4}{5}$ and $\frac{6}{7}$; also, $\frac{2}{3}$, $\frac{3}{7}$ and $\frac{4}{11}$.

Answers: $\frac{35}{70}, \frac{56}{70}, \frac{60}{70},$ and $\frac{154}{231}, \frac{99}{231}, \frac{84}{231}.$

(3) Express with a common denominator, the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{9}{13}$.

Answer: $\frac{195}{390}, \frac{260}{390}, \frac{234}{390}$ and $\frac{270}{390}.$

(4). Change the forms of $1\frac{1}{3}$, $2\frac{1}{5}$ and $3\frac{1}{7}$ into those of improper fractions with a common denominator.

Answer: $\frac{140}{105}, \frac{231}{105}$ and $\frac{330}{105}.$

(5) Reduce $\frac{4}{5}$, $2\frac{5}{9}$ and $3\frac{2}{11}$ to fractions having a common denominator.

Answer: $\frac{396}{495}, \frac{1265}{495}$ and $\frac{1575}{495}.$

(6) Put 7, $\frac{5}{8}$, $10\frac{10}{11}$ and $26\frac{1}{7}$ into fractions of the same denomination.

Answer: $\frac{4312}{616}, \frac{385}{616}, \frac{6720}{616}$ and $\frac{16104}{616}.$

(7) Transform $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{7}{8}$ into equivalent fractions, with the least common denominator.

Answer: $\frac{16}{24}, \frac{18}{24}$ and $\frac{21}{24}.$

(8) What are the fractions having the least common denominator, which are of equal values with $1\frac{4}{7}$, $\frac{19}{21}$ and $\frac{32}{35}$?

Answer: $\frac{165}{105}$, $\frac{95}{105}$ and $\frac{96}{105}$.

(9) Reduce $\frac{1}{3}$, $\frac{2}{9}$, $\frac{5}{12}$ and $\frac{7}{18}$ to the least common denominator.

Answer: $\frac{12}{36}$, $\frac{8}{36}$, $\frac{15}{36}$ and $\frac{14}{36}$.

(10) Exhibit $\frac{3}{7}$, $\frac{9}{14}$, $\frac{17}{21}$ and $\frac{25}{28}$, as fractions having the least common denominator.

Answer: $\frac{36}{84}$, $\frac{54}{84}$, $\frac{68}{84}$ and $\frac{75}{84}$.

(11) Reduce $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{21}$ and $\frac{1}{60}$, so as to have the least common denominator.

Answer: $\frac{140}{1680}$, $\frac{105}{1680}$, $\frac{80}{1680}$ and $\frac{28}{1680}$.

(12) Represent $1\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{1}{4}$, and $6\frac{1}{6}$ in the fewest figures, so as to have a common denominator.

Answer: $\frac{18}{12}$, $\frac{40}{12}$, $\frac{51}{12}$ and $\frac{74}{12}$.

(13) Express with the least common denominator, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ and $\frac{5}{6}$.

Answer: $\frac{30}{60}$, $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$ and $\frac{50}{60}$.

(14) Reduce $\frac{2\frac{1}{3}}{7}$, $8\frac{1}{4}$, $\dfrac{9+\frac{1}{11}}{9\times\frac{1}{11}}$ and $16\frac{17}{18}$, to fractions with the least common denominator.

Answer: $\frac{12}{36}$, $\frac{297}{36}$, $\frac{400}{36}$ and $\frac{610}{36}$.

(15) Find the greatest and least of the fractions, $\frac{4}{5}$, $\frac{5}{6}$ and $\frac{4+5}{5+6}$: also, of $\frac{4}{3}$, $\frac{5}{4}$, $\frac{9}{8}$ and $\frac{15}{14}$.

(16) Of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{9}$, $\frac{9}{10}$, find the greatest and least.

(17) Compare the quantities $2\frac{1}{2}$, $\frac{2}{7}$ of $9\frac{2}{5}$ and $\frac{7\frac{1}{2}}{2\frac{5}{6}}$.

I. ADDITION OF FRACTIONS.

86. **Rule.** Express the fractions with a common denominator; add together the new numerators, and under their sum place the common denominator: and the resulting fraction, reduced if possible, will be the sum required.

For, let $7\frac{2}{7}$ and $4\frac{7}{8}$ be the proposed quantities, which reduced to improper fractions are $\frac{51}{7}$ and $\frac{39}{8}$: then, since addition can be performed only upon quantities of the *same* denominations, these fractions must first be reduced to a *common* denominator; and their sum will be

$$\frac{51}{7}+\frac{39}{8}=\frac{408}{56}+\frac{273}{56}=\frac{681}{56}=12\tfrac{9}{56}.$$

The process in this case may be simplified: for,
the sum of the integers $= 7 + 4 = 11$:

$$\text{the sum of the fractions} = \frac{2}{7}+\frac{7}{8}=\frac{16+49}{56}=\frac{65}{56}=1\tfrac{9}{56}:$$

$$\text{and therefore the entire sum} = 11 + 1\tfrac{9}{56} = 12\tfrac{9}{56},$$

as before; and this is much shorter and easier, particularly when the numbers are large: but attention to the following examples worked out, will suggest other *abbreviations* and *forms* of practical importance.

Ex. What is the sum of $\frac{31}{7}$ and $\frac{47}{8}$: of $\frac{3}{4}$, $1\frac{4}{5}$ and $2\frac{5}{6}$: of $\frac{2}{3}$ of $3\frac{3}{10}$, $\frac{1\frac{3}{4}}{2\frac{5}{6}}$ of 17 and $\frac{3}{5}$ of $5\frac{3}{4}$ of $\frac{34}{51}$?

(1) The sum $=\frac{31}{7}+\frac{47}{8}=4\frac{3}{7}+5\frac{7}{8}=4+5+\frac{3}{7}+\frac{7}{8}$

$$=9+\frac{24}{56}+\frac{49}{56}=9+\frac{73}{56}=9+1\tfrac{17}{56}=10\tfrac{17}{56}.$$

(2) The sum $= \frac{3}{4} + 1\frac{4}{5} + 2\frac{5}{6} = 3 + \frac{3}{4} + \frac{5}{6} + \frac{4}{5}$

$$= 3 + \frac{18}{24} + \frac{20}{24} + \frac{4}{5} = 3 + 1\tfrac{7}{12} + \frac{4}{5} = 4 + 1\tfrac{23}{60} = 5\tfrac{23}{60}$$

(3) First, $\frac{2}{3}$ of $3\frac{3}{10} = \frac{2}{3}$ of $\frac{33}{10} = \frac{1}{1}$ of $\frac{11}{5} = \frac{11}{5}$:

$$\text{and } \frac{1\frac{3}{4}}{2\frac{5}{6}} = \frac{\frac{7}{4} \times 4 \times 6}{\frac{17}{6} \times 6 \times 4} = \frac{7 \times 6}{17 \times 4} = \frac{7 \times 3}{17 \times 2} = \frac{21}{34},$$

therefore $\frac{1\frac{3}{4}}{2\frac{5}{6}}$ of $17 = \frac{21}{34}$ of $17 = \frac{21}{2}$:

also, $\frac{3}{5}$ of $5\frac{3}{4}$ of $\frac{34}{51} = \frac{3}{5}$ of $\frac{23}{4}$ of $\frac{2}{3} = \frac{23}{10}$:

whence, the required sum will be

$$\frac{11}{5} + \frac{21}{2} + \frac{23}{10} = \frac{22}{10} + \frac{105}{10} + \frac{23}{10} = \frac{150}{10} = 15.$$

Examples for Practice.

(1) Find the sum of $\frac{2}{5}$ and $\frac{4}{7}$; of $\frac{3}{7}$ and $\frac{5}{9}$; of $\frac{3}{8}$ and $\frac{7}{9}$, and of $\frac{5}{8}$ and $\frac{12}{17}$.

Answers: $\frac{34}{35}$, $\frac{62}{63}$, $1\frac{11}{72}$ and $1\frac{45}{136}$.

(2) Add together $1\frac{1}{2}$ and $7\frac{1}{5}$; $2\frac{6}{7}$ and $13\frac{3}{10}$; $5\frac{1}{6}$ and $12\frac{4}{5}$, and $37\frac{2}{11}$ and $24\frac{12}{13}$.

Answers: $8\frac{7}{10}$, $16\frac{11}{70}$, $17\frac{29}{30}$ and $62\frac{15}{143}$.

(3) What is the sum of $\frac{41}{18}$ and $\frac{21}{13}$; of $\frac{57}{5}$ and $\frac{38}{15}$; of $\frac{31}{9}$ and $\frac{49}{8}$; of $\frac{27}{16}$ and $\frac{71}{24}$?

Answers: $3\frac{209}{234}$, $13\frac{14}{15}$, $9\frac{41}{72}$, $4\frac{31}{48}$.

(4) Add together $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{5}{6}$; $\frac{1}{7}$, $\frac{2}{5}$ and $\frac{3}{11}$; $\frac{4}{5}$, $\frac{6}{7}$ and $\frac{9}{10}$; $\frac{5}{6}$, $\frac{4}{9}$ and $\frac{14}{21}$.

Answers: $2\frac{1}{4}$, $\frac{314}{385}$, $2\frac{39}{70}$, $1\frac{17}{18}$.

(5) Add together $\frac{11}{16}$, $\frac{45}{8}$, $\frac{97}{2}$; $2\frac{1}{5}$, $3\frac{2}{5}$, $5\frac{3}{8}$; $8\frac{4}{7}$, $13\frac{5}{9}$, $27\frac{8}{11}$; $\frac{15}{4}$, $3\frac{1}{3}$, $11\frac{14}{5}$.

Answers: $54\frac{13}{16}$, $10\frac{287}{300}$, $49\frac{592}{693}$, $20\frac{53}{60}$.

(6) Find the sum of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$; of $\frac{1}{4}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$; of $\frac{1}{40}$, $\frac{7}{20}$, $\frac{6}{5}$, $\frac{1}{8}$; of $\frac{3}{7}$, $\frac{5}{14}$, $\frac{11}{42}$, $\frac{20}{21}$.

Answers: $3\frac{1}{8}$, $1\frac{659}{1260}$, $1\frac{7}{10}$, 2.

(7) Find the sum of $\frac{3}{13}$ of $9\frac{8}{15}$, and $\frac{4}{17}$ of $8\frac{1}{2}$.

Answer: $4\frac{1}{5}$.

(8) Required the sum of $14\frac{3}{4}$ and $\frac{2}{3}$ of $\frac{5}{6}$ of 8: of $\frac{2}{7}$, $4\frac{1}{3}$ and $\frac{3}{5}$ of 2: of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{8}$, $\frac{1}{5}$ of $\frac{2}{3}$ of $\frac{15}{8}$, $\frac{3}{19}$ of $\frac{1}{9}$ of $\dfrac{28\frac{1}{2}}{2}$: of $\frac{2}{3}$ of $\frac{5}{7}$, 9, $\dfrac{2\frac{4}{5}}{7}$, $\dfrac{1\frac{2}{3}}{2\frac{1}{2}}$.

Answers: $19\frac{7}{36}$, $5\frac{86}{105}$, $\frac{15}{16}$, $10\frac{19}{35}$.

(9) Required the value of $1\frac{1}{3} + \dfrac{8}{3}$ of $\dfrac{41}{34} + \dfrac{4}{5\frac{1}{10}}$.

Answer: $5\frac{1}{3}$.

(10) Determine the quantity in its simplest form which shall be equivalent to the sum of the magnitudes,

$\frac{2}{3}$ of $\frac{3}{5}$ of $11\frac{1}{2}$, $1\frac{3}{4}$ of $3\frac{5}{6}$ of $\frac{1}{23}$, $2\frac{5}{7}$ of $4\frac{2}{3}$ of $\frac{1}{152}$.

Answer: $4\frac{39}{40}$.

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(11) Find the respective sums of $1\frac{2}{7}$, $\frac{5}{9}$, $\frac{8}{11}$, $3\frac{1}{5}$: of $3\frac{5}{8}$, $2\frac{3}{5}$, $\frac{7}{11}$, $7\frac{1}{6}$: and of $2\frac{7}{12}$, $3\frac{5}{16}$, $4\frac{8}{21}$, $5\frac{11}{60}$.

Answers: $5\frac{2663}{3465}$, $14\frac{37}{1320}$ and $15\frac{773}{1680}$.

(12) Add together $\frac{2}{5}$, $\frac{35}{80}$, $\frac{14}{100}$, $\frac{3}{140}$, $\frac{3}{2800}$: also, $387\frac{1}{2}$, $285\frac{1}{4}$, $394\frac{1}{3}$, $\frac{2}{5}$ of 3704: and $3\frac{5}{12}$, $7\frac{1}{2}$, $8\frac{2}{3}$, $4\frac{1}{6}$.

Answers: 1, $2548\frac{41}{60}$ and $23\frac{3}{4}$.

(13) Add together $3\frac{1}{2}$, $4\frac{1}{3}$, $5\frac{1}{4}$, $\frac{3}{4}$ of $\frac{7}{8}$, $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{5}{8}$: $\frac{4}{11}$ of $\frac{2}{3}$ of $\frac{1}{2}$, $\frac{7}{8}$ of $\frac{4}{5}$ of $\frac{10}{33}$, $\frac{2}{5}$ of $1\frac{2}{3}$, $2\frac{5}{7}$: and $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{9}{20}$ of $11\frac{1}{9}$.

Answers: $13\frac{27}{32}$, $3\frac{5}{7}$ and $8\frac{11}{20}$.

(14) Find the sum of $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{4}{9}$, $\frac{7}{12}$, $\frac{11}{18}$, $\frac{13}{24}$; and of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{7}{15}$, $\frac{23}{30}$, $\frac{11}{40}$, $\frac{23}{60}$, $\frac{41}{120}$.

Answers: $3\frac{8}{9}$ and 4.

(15) Shew that the simple fraction equivalent to the value of $\frac{1}{3}$ of $\dfrac{\frac{1}{2}}{\frac{5}{4}} + \frac{1}{2}$ of $\dfrac{\frac{3}{4}}{\frac{9}{16}+1} + \frac{1}{6}$ of $\dfrac{\frac{1}{5}}{4}$, is of the same magnitude as that expressed by $\frac{3}{10} + \frac{1}{2}$ of $\frac{7}{15}$ of $\frac{7}{20}$.

II. SUBTRACTION OF FRACTIONS.

87. Rule. Reduce the fractions to a common denominator; subtract the less numerator from the greater; under the remainder place the common denominator, and the result, properly reduced, will be the required difference.

For, taking the quantities $5\frac{1}{3}$ and $2\frac{4}{7}$, and reducing them to fractional forms, we have the difference

$$= \frac{16}{3} - \frac{18}{7} = \frac{112}{21} - \frac{54}{21} = \frac{58}{21} = 2\tfrac{16}{21}.$$

This operation may be performed in a more convenient form as follows:

the difference $= 5\frac{1}{3} - 2\frac{4}{7} = 5\frac{7}{21} - 2\frac{12}{21} = 2\frac{16}{21}$: where $\frac{12}{21}$, being greater than $\frac{7}{21}$, is subtracted from $\dfrac{7}{21} + 1$ or $\dfrac{28}{21}$, and 1 is carried to the whole number 2, as in Integers.

Examples for Practice.

(1) Find the difference of $\frac{3}{5}$ and $\frac{1}{6}$; of $\frac{7}{9}$ and $\frac{3}{7}$; of $\frac{2}{9}$ and $\frac{3}{11}$; of $\frac{5}{6}$ and $\frac{11}{15}$; of $1\frac{3}{28}$ and $1\frac{2}{21}$.

Answers: $\dfrac{13}{30}, \dfrac{22}{63}, \dfrac{5}{99}, \dfrac{3}{10}, \dfrac{1}{84}$.

(2) What is the difference of $19\frac{2}{7}$ and $13\frac{2}{11}$; of $8\frac{12}{25}$ and $17\frac{13}{27}$; of 1000 and $384\frac{7}{39}$; of $279\frac{2}{7}$ and $168\frac{3}{5}$?

Answers: $6\frac{8}{77}$, $9\frac{1}{675}$, $615\frac{32}{39}$, $110\frac{24}{35}$.

(3) Required the difference of $1\frac{2}{3}$ of $3\frac{3}{8}$ and $2\frac{7}{8}$ of $16\frac{2}{7}$; of $\frac{3}{20}$ of $\frac{5}{9}$ and $\frac{3}{7}$ of $\frac{2}{11}$ of 25: of $2\frac{1}{2}$ of $4\frac{1}{4}$ and $7\frac{1}{7}$ of $10\frac{1}{9}$.

Answers: $39\frac{11}{56}$, $1\frac{799}{924}$, $61\frac{43}{72}$.

(4) Find the difference of $\frac{3}{5}$ of $\frac{4\frac{1}{6}}{5\frac{1}{7}}$ and $\frac{2}{3}$ of $\frac{15}{2}$: of $\frac{3\frac{2}{5}}{4\frac{7}{9}}$ and $\frac{6\frac{2}{3}}{12\frac{2}{7}}$: of $2\frac{3}{5}$ of $\frac{5\frac{1}{4}}{4\frac{1}{5}}$ and $\frac{7\frac{1}{3}}{11}$ of $15\frac{3}{8}$.

Answers: $4\frac{37}{72}$, $\frac{100}{645}$, 7.

(5) Prove that the sum of $5\frac{1}{3}$ and $3\frac{1}{5}$, is equal to four times their difference.

III. MULTIPLICATION OF FRACTIONS.

88. Rule. Multiply together the respective numerators and denominators, reduced to fractional forms if necessary; and the fraction thence arising will be the product, which may generally be simplified by means of the preceding Articles.

For, let the fractions be $\frac{2}{9}$ and $\frac{7}{8}$; then if $\frac{2}{9}$ be multiplied by 7, the product will be $\frac{14}{9}$ by Article (74): but 7 being 8 times as great as $\frac{7}{8}$, the multiplier above used is 8 times too *large*, and the product $\frac{14}{9}$ will therefore be 8 times too *large* also: whence, the product required must be

$$\frac{14}{9} \div 8 = \frac{14}{72} = \frac{7}{36}:$$

or, the product $= \frac{2}{9} \times \frac{7}{8} = \frac{2 \times 7}{9 \times 8} = \frac{14}{72} = \frac{7}{36}$.

89 If three or more fractions be proposed, as $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$; their continued product is $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$

$$= \frac{1}{2} \times \text{the product of } \frac{2}{3} \text{ and } \frac{3}{4} = \frac{1}{2} \times \frac{6}{12} = \frac{6}{24} = \frac{1}{4};$$

$$\text{or, } \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4};$$

and thus the rule may be proved to be general: also, in cases like this, the reduction is shortened by *cancelling* from the products of the numerators and denominators, any factor or factors common to both, and effecting the multiplications of what are left; thus,

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{1 \times 1 \times 1}{1 \times 1 \times 4} = \frac{1}{4}, \text{ as above.}$$

Hence, the *product* of two or more fractions is the value of the *compound* fraction of which they are the *constituent* fractions: and thus, *Multiplication* is here extended so as to express the *part* or *parts* of any quantity: for, $\frac{5}{6}$ of $\frac{7}{8} = \frac{5 \times 7}{6 \times 8} = \frac{5}{6} \times \frac{7}{8}$, in the same way as $\frac{5}{6}$ of 7, 5 of $\frac{7}{8}$ and 5 of 7 may be replaced by $\frac{5}{6} \times 7$, $5 \times \frac{7}{8}$ and 5×7, which are $\frac{35}{6}$, $\frac{35}{8}$ and 35 respectively.

Examples for Practice.

(1) Required the product of $\frac{2}{5}$ and $\frac{3}{7}$; of $\frac{3}{8}$ and $\frac{5}{9}$; of $2\frac{3}{5}$ and $7\frac{2}{9}$; of $8\frac{2}{7}$ and $10\frac{5}{11}$; of $6\frac{3}{8}$ and $14\frac{6}{17}$.

Answers: $\frac{6}{35}$, $\frac{5}{24}$, $18\frac{7}{9}$, $86\frac{48}{77}$, $91\frac{1}{2}$.

(2) Find the product of $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{7}{12}$: of $\frac{3}{8}$, $\frac{11}{7}$ and $\frac{15}{11}$: of $\frac{49}{133}$, $1\frac{1}{75}$ and $\frac{28}{98}$: of $\frac{428}{515}$, $\frac{5253}{1819}$ and $\frac{5}{4}$: of $2\frac{3}{5}$, $\frac{6\frac{1}{4}}{1\frac{3}{4}}$ and $\frac{3\frac{2}{5}}{5\frac{1}{7}}$.

Answers: $\frac{7}{30}$, $\frac{45}{56}$, $\frac{8}{75}$, 3, $5\frac{2}{8}$.

(3) Required the product of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$: of $\frac{3}{4}$, $\frac{6}{7}$, $\frac{8}{9}$ and $\frac{10}{11}$: of $\frac{8}{13}$, $2\frac{2}{3}$, $1\frac{1}{25}$ and $1\frac{11}{64}$: of $6\frac{2}{5}$, $9\frac{5}{8}$, $12\frac{6}{11}$ and $\frac{1\frac{1}{2}}{112}$: of $\frac{5}{6}$, $2\frac{3}{4}$, $3\frac{5}{11}$, $5\frac{2}{19}$ and $6\frac{1}{194}$.

Answers: $\frac{1}{5}$, $\frac{40}{77}$, 2, $10\frac{1}{2}$, $242\frac{17}{24}$.

(4) Multiply $2\frac{3}{8}$ by $\frac{1}{3}$ of $\frac{2}{5}$ of $\frac{7}{9}$: $13\frac{5}{9}$ of $7\frac{1}{3}$ by $\frac{3}{4}$ of $\frac{4}{9}$ of $12\frac{1}{2}$: $\frac{2}{3\frac{3}{16}}$ of $\frac{6\frac{1}{11}}{8}$ by $\frac{2}{3}$ of $8\frac{1}{4}$ of $\frac{35}{66}$.

Answers: $\frac{133}{540}$, $414\frac{16}{81}$, $1\frac{7}{18}$.

(5) Multiply together $\frac{2\frac{3}{4}}{5\frac{7}{9}}$ of $\frac{1}{3}$ and $\frac{3}{5}$ of $\frac{4\frac{3}{8}}{7\frac{1}{2}}$ of $\frac{7\frac{3}{7}}{5\frac{1}{5}}$: $\frac{11\frac{3}{5}}{29}$ of $\frac{4\frac{1}{3}}{13}$ of $\frac{3\frac{1}{4}}{10\frac{3}{8}}$ and $5\frac{1}{5}$ of 6 of $20\frac{3}{4}$.

Answers: $\frac{33}{416}$, 26.

(6) Find the continued product of the fractions, $\frac{324}{361}$, $\frac{1444}{1296}$, $\frac{441}{529}$ and $\frac{2116}{1764}$.

Answer: 1.

IV. DIVISION OF FRACTIONS.

90. Rule. Multiply the dividend by the divisor *inverted*, and the result will be the quotient, which may be reduced, when possible: or, which is the same thing, *invert* the divisor, and then proceed by the rule for the Multiplication of Fractions.

For, let $\frac{3}{7}$ be to be divided by $\frac{4}{5}$; then $\frac{3}{7} \div 4 = \frac{3}{28}$ is 5 times too *small*, because the divisor has been taken 5 times too *great*: whence, the quotient will be

$$\frac{3}{28} \times 5 = \frac{15}{28}:$$

$$\text{or, the quotient is } \frac{15}{28} = \frac{3 \times 5}{7 \times 4} = \frac{3}{7} \times \frac{5}{4};$$

and the operation may be expressed in the following *form*:

$$\text{the quotient} = \frac{3}{7} \div \frac{4}{5} = \frac{3}{7} \times \frac{5}{4} = \frac{15}{28}.$$

Hence, *Division* is here extended to express the finding of the fraction, the product of which and the divisor is the dividend: and the *quotient* shews what *part* or *parts* the dividend is of the divisor.

91. To denote the division of one integer by another, as of 4 by 5, we have the quotient

$$= \frac{4}{1} \div \frac{5}{1} = \frac{4}{1} \times \frac{1}{5} = \frac{4}{5};$$

or, in *words*, a simple fraction may be considered as an adequate expression of the *implied* division of the numerator by the denominator: and therefore the fraction $\frac{4}{5}$, in addition to its meaning as explained in Article (71), implies that if 4 were divided into *five* equal parts, *one* of these parts is expressed by it.

Examples for Practice.

(1) Find the quotient of $\frac{2}{7}$ by $\frac{3}{8}$: of $\frac{3}{5}$ by $\frac{4}{9}$: of $\frac{4}{11}$ by $\frac{6}{13}$: of $\frac{19}{5}$ by $\frac{83}{10}$: of $\frac{26}{51}$ by $\frac{91}{34}$.

Answers: $\frac{16}{21}$, $1\frac{7}{20}$, $\frac{26}{33}$, $\frac{38}{83}$, $\frac{4}{21}$.

(2) What is the quotient of $2\frac{1}{5}$ by $3\frac{2}{9}$: of $10\frac{4}{9}$ by $13\frac{3}{7}$: of $17\frac{2}{3}$ by $7\frac{13}{14}$: of $\frac{7\frac{3}{7}}{40\frac{5}{9}}$ by $\frac{17\frac{1}{3}}{73}$?

Answers: $\frac{99}{145}$, $\frac{7}{9}$, $2\frac{76}{333}$, $\frac{27}{35}$.

(3) Divide $3\frac{4}{9}$ by $\frac{1}{2}$ of $\frac{4}{7}$ of $\frac{5}{6}$: $3\frac{1}{125}$ of $3\frac{4}{7}$ by $\frac{47}{315}$ of 9: $15\frac{7}{11}$ of $8\frac{4}{5}$ by $\frac{4}{5}$ of $\frac{6}{11}$ of $15\frac{5}{8}$: $2\frac{10}{11}$ of $5\frac{1}{18}$ of 1351 by $3\frac{11}{12}$ of $\frac{44}{13\frac{4}{7}}$ of $202\frac{1}{2}$.

Answers: $14\frac{7}{15}$, 8, $19\frac{67}{95}$, $7\frac{10}{45}$.

(4) Compare the product and quotient of $\frac{7}{9}$ by $\frac{10}{11}$.

Answer: $\frac{700}{990}$ and $\frac{847}{990}$.

92. What has been proved in the adaptation of the fundamental operations to fractions, will furnish the means of simplifying arithmetical expressions formed by their combinations: and, in general, only very slight *mental* exertion will be required, if the attention of the *eye* be directed to the *composition* of the *terms* of the fractions concerned, and their *resolution* into the *factors* of which they are made up.

(1) The value of $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}=\left(\frac{1}{2}+\frac{1}{4}\right)-\left(\frac{1}{3}+\frac{1}{5}\right)$

$$=\frac{3}{4}-\frac{8}{15}=\frac{45}{60}-\frac{32}{60}=\frac{13}{60}.$$

(2) The value of $\left(\frac{1}{3}+\frac{1}{5}\right)\times\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{8}{15}\times\frac{5}{14}=\frac{4}{21}$.

(3) The value of $\left(\frac{4}{7}-\frac{2}{11}\right)\div\left(\frac{5}{6}+\frac{3}{8}\right)=\frac{30}{77}\div\frac{58}{48}$

$$=\frac{30\times 48}{77\times 58}=\frac{30\times 24}{77\times 29}=\frac{720}{2233}.$$

(4) To find the value of $\left\{2\tfrac{3}{4} + \tfrac{5}{2} \text{ of } \dfrac{7}{3\frac{4}{5}} - \dfrac{1\frac{2}{3}}{2\frac{1}{2}}\right\} \div 1\tfrac{77}{228}$;

we have $2\tfrac{3}{4} = \dfrac{11}{4}$: $\tfrac{5}{2}$ of $\dfrac{7}{3\frac{4}{5}} = \tfrac{5}{2}$ of $\dfrac{35}{19} = \dfrac{175}{38}$:

$\dfrac{1\frac{2}{3}}{2\frac{1}{2}} = \dfrac{\frac{5}{3}}{\frac{5}{2}} = \dfrac{2}{3}$: whence, the quantity within the brackets

$$= \frac{11}{4} + \frac{175}{38} - \frac{2}{3} = \frac{11}{4} - \frac{2}{3} + \frac{175}{38} = \frac{25}{12} + \frac{175}{38}$$

$$= \frac{25}{2} \times \left\{\frac{1}{6} + \frac{7}{19}\right\} = \frac{25 \times 61}{228}: \text{ also, } 1\tfrac{77}{228} = \frac{305}{228};$$

therefore, the value required is $\dfrac{25 \times 61}{228} \times \dfrac{228}{305} = 5$.

Examples for Practice.

(1) Required the value of $\dfrac{5}{6} - \dfrac{3}{4} + \dfrac{2}{3} - \dfrac{1}{2}$, and of

$$\frac{15}{16} - \frac{14}{15} + \frac{13}{14} - \frac{11}{12}.$$

Answers: $\dfrac{1}{4}$ and $\dfrac{9}{560}$.

(2) Reduce to their simplest forms, the expressions,

$$\frac{1}{2} + \frac{2}{3} - \frac{1}{6} + \frac{3}{8} - \frac{1}{12},$$

and $\dfrac{1}{2} - \dfrac{3}{8} + \dfrac{7}{15} - \dfrac{11}{40} + \dfrac{5}{6} + \dfrac{2}{3} - \dfrac{5}{12} + \dfrac{3}{5}$.

Answers: $1\tfrac{7}{24}$ and 2.

(3) Multiply the sum of $\dfrac{1}{2}$, $1\tfrac{3}{7}$ and $\dfrac{5}{6}$ by the difference of $\dfrac{4}{15}$ and $\dfrac{3}{20}$; and divide the product by $\dfrac{11}{18}$ of $1\tfrac{14}{15}$.

Answer: $\dfrac{3}{11}$.

(4) Reduce $\dfrac{1}{2}$ of $(6\tfrac{1}{2} + 2\tfrac{2}{3} - 3)$, and $\left(\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}\right) \times \left(\dfrac{1}{5} \text{ of } \dfrac{2}{7}\right)$ to their simplest forms.

Answers: $3\tfrac{1}{12}$ and $\dfrac{13}{210}$.

(5) Simplify as much as possible, the arithmetical expression, $\left(\frac{1}{2} \times \frac{3}{8} + \frac{3}{7} \times \frac{2}{5}\right) - \left(\frac{2}{7} \times \frac{5}{9} - \frac{1}{8} \times \frac{4}{11}\right)$.

Answer: $\frac{13619}{55440}$.

(6) Determine the simple fraction which expresses the value of $\left(\frac{5}{7} \times \frac{2}{9} \times 13\frac{1}{2}\right) \div \left(\frac{1}{9} \times \frac{3}{7} + 54\right)$.

Answer: $\frac{9}{227}$.

(7) What is the value of the expression,

$$\frac{2247}{1017} \div \frac{903}{1107} \times \frac{774}{615} \div \frac{1926}{565}?$$

Answer: 1.

(8) Required the value of the expression,

$$\frac{3}{8} \text{ of } \frac{4}{7} - \frac{2}{11} \text{ of } 3\frac{1}{7} + \frac{5}{9} \text{ of } 3\frac{3}{8}.$$

Answer: $1\frac{29}{56}$.

(9) Simplify $\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \div \left(\frac{1}{2\frac{1}{2}} + \frac{1}{3\frac{1}{2}} + \frac{1}{4\frac{1}{2}}\right) - \frac{13}{24}$ of $\frac{576}{264}$.

Answer: $\frac{1}{88}$.

(10) Prove that the value of the expression,

$$\frac{1}{7} \text{ of } \left(1 - \frac{2}{7}\right) + \frac{4}{5} \text{ of } \frac{1}{10} + \frac{3}{5} \text{ of } \left(\frac{1}{2} + \frac{11}{14}\right) + \frac{3}{70} \text{ of } \left(\frac{2}{7} + \frac{4}{5}\right)$$

is 1, by the simplest process.

REDUCTION OF FRACTIONS.

93. Our attention has hitherto been confined to fractions considered *generally*, without regard to the particular species of their *units;* and it remains to apply what has been said to such *concrete* quantities as constitute the principal subjects of practical computation.

94. *A Fraction may be transformed into another, so that the value of the unit in the latter may have a specified relation to that of the unit in the former.*

RULE. *Multiply* or *divide* the fraction by the numbers which connect the different denominations in order, according as the value of the unit in the required fraction, is *less* or *greater* than that of the unit in the one which is given.

For, let the fraction be £$\frac{2}{7}$, where the unit is one *pound:* then if it be required to find the fraction when the unit is one *farthing*, it is manifest from what has been said in the Reduction of compound quantities, that in order to retain the same *absolute* value, we must have $20 \times 12 \times 4$ times as great a *fraction* as the original one: that is,

$$\overset{\text{£.}}{\frac{2}{7}} = \frac{2}{7} \times \overset{\text{farthings.}}{\frac{20}{1} \times \frac{12}{1}} \times \frac{4}{1} = \overset{\text{far.}}{\frac{1920}{7}};$$

and the value of the unit in the latter fraction being $\frac{1}{960}$th part of that in the former, the same *absolute value* is retained by taking 960 times as many parts in the latter, as in the former.

Again, reversing the operation, we have

$$\overset{\text{far.}}{\frac{1920}{7}} = \frac{1920}{7} \times \overset{\text{£.}}{\frac{1}{4}} \times \frac{1}{12} \times \frac{1}{20} = \overset{\text{£.}}{\frac{1920}{6720}} = \overset{\text{£.}}{\frac{2}{7}};$$

the divisors 4, 12 and 20 being *inverted*, according to the rule laid down for the Division of Fractions.

Ex. Find what fraction of a crown, is equivalent to $\frac{1}{4}$ of a pound.

We have here $\overset{\text{£.}}{\frac{1}{4}} = \overset{\text{s.}}{\frac{1}{4}} \times \frac{20}{1} = \overset{\text{s.}}{\frac{20}{4}} = \overset{\text{crowns.}}{\frac{20}{4} \times \frac{1}{5}} = \overset{\text{cr.}}{\frac{20}{20}} = \overset{\text{cr.}}{\frac{1}{1}}$:

and we know very well that $\frac{1}{4}$ of £1, or 5*s*., is equal to 1 crown, expressed fractionally by $\frac{1}{1}$ cr.

Examples for Practice.

(1) Reduce $\frac{1}{7}$, $\frac{4}{9}$ and $\frac{3}{64}$ of a pound, to fractions of a penny.

Answers: $\frac{240}{7}$, $\frac{320}{3}$ and $\frac{45}{4}$.

(2) Express $\frac{2}{5}$ of a shilling, $\frac{5}{7}$ of a penny and $\frac{160}{11}$ of a farthing, as fractions of a pound.

$$\text{Answers: } \frac{1}{50}, \frac{1}{336} \text{ and } \frac{1}{66}.$$

(3) Reduce $\frac{2}{9}$ of a guinea, $\frac{3}{4}$ of a half-guinea and $6\frac{6}{7}$ of a crown, to fractions of £1.

$$\text{Answers: } \frac{7}{30}, \frac{63}{160} \text{ and } \frac{12}{7}.$$

(4) Reduce $\frac{3}{32}$ of a cwt. to the fraction of 1 lb, and $\frac{4}{7}$ of an ounce to that of 1 cwt.

$$\text{Answers: } \frac{21}{2} \text{ and } \frac{1}{3136}.$$

(5) Express $\frac{7}{342}$ of a yard as the fraction of an inch, and $\frac{108}{145}$ of an inch as that of a pole.

$$\text{Answers: } \frac{14}{19} \text{ and } \frac{6}{1595}.$$

(6) Find the fraction of a yard which expresses $\frac{3}{4}$ of an ell of 5 quarters; and that of a day which is equivalent to $\frac{5}{146}$ of a year of 365 days.

$$\text{Answers: } \frac{15}{16} \text{ and } \frac{25}{2}.$$

(7) Reduce $\frac{4}{279}$ of a barrel of beer to the fraction of a quart; and $\frac{4}{11}$ of a pint of wine, to the fraction of a hogshead.

$$\text{Answers: } \frac{64}{31} \text{ and } \frac{1}{1386}.$$

(8) Required the fractions of £10., which are equivalent to $\frac{4}{7}$ of a guinea, $\frac{2}{9}$ of a shilling, and $\frac{16}{15}$ of a farthing.

$$\text{Answers: } \frac{3}{50}, \frac{1}{900} \text{ and } \frac{1}{9000}.$$

(9) Express $\frac{3}{4}$ of $\frac{5}{7}$ of 1 qr. as the fraction of a ton, and $4\frac{1}{2}$ of $2\frac{1}{7}$ of a square inch as the fraction of a square yard.

$$\text{Answers: } \frac{3}{448} \text{ and } \frac{5}{672}.$$

(10) What are the fractions which express $\frac{1}{3}$ of $2151\frac{1}{9}$ square yards in acres, and $\dfrac{5\frac{1}{12}}{7\frac{1}{14}}$ of 5940 seconds in weeks?

$$\text{Answers: } \frac{4}{27} \text{ and } \frac{61}{8640}.$$

95. *The Value of a compound quantity may be exhibited in the form of a fraction, whereof the unit is of a specified denomination.*

Rule. Reduce the proposed quantity to the lowest denomination contained in it, and also the proposed unit to the same denomination; then the fraction whose numerator and denominator are these results respectively, will be the one required.

For, let it be required to represent 2qrs. 15lbs. as the fraction of 1cwt.: then we have

qrs. lbs.	cwt.
2 . 15	1
28	4
71 lbs.	4
	28
	112 lbs.

and of the 112 equal parts into which 1cwt. is supposed to be divided, 71 are here taken, so that according to Article (71), the fraction required will be $\frac{71}{112}$cwt.

96. By the two preceding rules, magnitudes of the same kind, consisting of fractions of simple or compound quantities, and connected by the operations of addition or subtraction, may be reduced to simple fractions of a given denomination.

Ex. Find the fraction of £1, which is equivalent to the excess of $\frac{2}{3}$ of a guinea, above the sum of $\frac{3}{4}$ of a shilling and $\frac{4}{9}$ of 7*s.* 6*d.*

$$\text{Here } \overset{g.}{\frac{2}{3}} = \overset{£.}{\frac{2}{3} \times \frac{21}{20}} = \overset{£.}{\frac{7}{10}}:$$

$$\overset{s.}{\frac{3}{4}} = \overset{£.}{\frac{3}{4} \times \frac{1}{20}} = \overset{£.}{\frac{3}{80}}:$$

$$\frac{4}{9} \text{ of } 7s.\ 6d. = \frac{4}{9} \text{ of } \frac{3}{8} = \overset{£.}{\frac{1}{6}}:$$

therefore the fraction $= \frac{7}{10} - \frac{3}{80} - \frac{1}{6} = \frac{119}{240}$.

Examples for Practice.

(1) Express 4*s.* 11*d.*; 17*s.* $11\frac{1}{2}$*d.*; 19*s.* $10\frac{3}{4}$*d.* and £1. 13*s.* $11\frac{1}{2}$*d.* $\frac{10}{21}$*f.*, as fractions of £1.

Answers: $\frac{59}{240}$, $\frac{431}{480}$, $\frac{191}{192}$ and $\frac{107}{63}$.

(2) What fraction is 2cwt. 1qr. 16lbs. of a ton; 2ft. 9in. of a pole, and 3ro. 25po. of an acre?

Answers: $\frac{67}{560}$, $\frac{1}{6}$ and $\frac{29}{32}$.

(3) Express 5bush. 3pks. 1gal. as the fraction of a quarter; and 2wks. 5days. 18hrs. as the fraction of a year of 365 days.

Answers: $\frac{47}{64}$ and $\frac{79}{1460}$.

(4) What fraction of £2. 7*s.* 6*d.* is £1. 7*s.* $8\frac{1}{2}$*d.*, and of 3cwt. 2qrs. 14lbs. is 3cwt. 19lbs. 2oz.?

Answers: $\frac{7}{12}$ and $\frac{2841}{3248}$.

(5) Reduce $\frac{2}{5}$ of 2*s.* $4\frac{1}{2}$*d.* to the fraction of a half crown; and 9*s.* $10\frac{1}{2}$*d.* to the fraction of 13*s.* $2\frac{1}{4}$*d.*

Answers: $\frac{19}{50}$ and $\frac{158}{211}$.

(6) Required the fractions of £1, which express the values of $\frac{3}{7}$ of $\frac{6\frac{2}{5}}{9\frac{1}{10}}$ of £1. 12*s.* $1\frac{1}{2}$*d.* and $\frac{5}{11}$ of £5. 17*s.* 4*d.*

Answers: $\frac{257}{540}$ and $\frac{8}{3}$.

(7) Find the simple fraction of £1. which expresses the sum of $\frac{1}{3}$ of $\frac{7}{10}$ of 13*s.* 4*d.* and $\frac{3}{4}$ of $\frac{5}{7}$ of 10*s.* 6*d.*

$$\text{Answer: } \frac{629}{1440}.$$

(8) Express the sum of $\frac{1}{9}$ of a guinea, $\frac{3}{8}$ of a pound, $\frac{5}{24}$ of a shilling and $\frac{1}{4}$ of a penny, as the fraction of a guinea; and the excess of $\frac{2}{11}$ of £13. 10*s.* 10$\frac{1}{2}$*d.* above $\frac{5}{7}$ of £1. 2*s.* 9*d.* as the fraction of £6.

$$\text{Answers: } \frac{23}{48} \text{ and } \frac{11}{40}.$$

(9) Compare the values of $\frac{1}{21}$ of a pound, $\frac{1}{22}$ of a guinea and $\frac{1}{4}$ of 3*s.* 9$\frac{1}{2}$*d.*

$$\text{Answer: } \frac{7040}{7392}, \frac{7056}{7392} \text{ and } \frac{7007}{7392}.$$

(10) Reduce $\dfrac{3\frac{1}{9}}{1\frac{1}{13}}$ of $\left\{\dfrac{19}{120} \text{ of } £1. - \dfrac{7}{48} \text{ of } 1s.\right\}$ to the fraction of a moidore of 27*s.*

$$\text{Answer: } \frac{1885}{5832}.$$

97. *If the Species of the unit be given, the value of a fraction of it may be expressed by means of its known parts.*

Rule. Multiply the numerator of the fraction by the number of parts of the next inferior denomination which are equivalent in value to the unit, divide the product by the denominator, and the quotient is the required number of parts of that denomination: proceed in the same way with the remainder, if any, and the parts of the next denomination will be found: and repeat this process till the lowest denomination, to which the unit is capable of being reduced, is obtained.

For, if the fraction be $\frac{5}{6}$ of a yard, we have

$$\overset{\text{yds.}}{\frac{5}{6}} = \overset{\text{feet.}}{\frac{5}{6}} \times \frac{3}{1} = \overset{\text{ft.}}{\frac{15}{6}} = \overset{\text{ft.}}{2\tfrac{1}{2}}:$$

$$\overset{\text{ft.}}{\frac{1}{2}} = \overset{\text{inches.}}{\frac{1}{2}} \times \frac{12}{1} = \overset{\text{in.}}{\frac{12}{2}} = \overset{\text{in.}}{6}:$$

and therefore the value of $\frac{5}{6}$ of a yard, expressed in the known parts of a yard, is 2ft. 6in., or 30 in.

98. The preceding Articles enable us to find the value of the sum or difference of fractional parts of magnitudes of the same kind.

Ex. Required the sum and difference of $\frac{2}{3}$ of a pound and $\frac{4}{9}$ of a guinea.

$$\text{Here, } \frac{2}{3} \text{ of a pound} = \frac{2}{3} \text{ of } \overset{s.}{20} = \overset{s.}{\frac{40}{3}} = \overset{s.}{13} \,.\, \overset{d.}{4}:$$

$$\frac{4}{9} \text{ of a guinea} = \frac{4}{9} \text{ of } 21 = \frac{84}{9} = 9 \,.\, 4:$$

$$\text{therefore the sum} = \overset{s.}{13} \,.\, \overset{d.}{4} + \overset{s.}{9} \,.\, \overset{d.}{4} = \overset{£.}{1} \,.\, \overset{s.}{2} \,.\, \overset{d.}{8}.$$

$$\text{and the difference} = 13 \,.\, 4 - 9 \,.\, 4 = 0 \,.\, 4 \,.\, 0.$$

The same results may also be obtained as follows:

$$\text{since } \overset{g.}{\frac{4}{9}} = \frac{4}{9} \times \overset{£.}{\frac{21}{20}} = \overset{£.}{\frac{7}{15}}, \text{ we have}$$

$$\text{the sum} = \frac{2}{3} + \frac{7}{15} = \frac{17}{15} = \overset{£.}{1} \,.\, \overset{s.}{2} \,.\, \overset{d.}{8}:$$

$$\text{the difference} = \frac{2}{3} - \frac{7}{15} = \frac{3}{15} = 0 \,.\, 4 \,.\, 0:$$

and when fractions of the unit of the *lowest* order occur, this will be the more convenient method of the two.

Examples for Practice.

(1) Find the values of $\frac{3}{5}$ of a pound, $\frac{5}{9}$ of a shilling and $\frac{5}{18}$ of a guinea.

Answers: 12*s.*; 6½*d.* ⅔*f.* and 5*s.* 10*d.*

(2) Required the values of $\frac{5}{7}$cwt., $\frac{5}{14}$qrs. and $\frac{3}{8}$lbs.

Answers: 2qrs. 24lbs.; 10lbs. and 6oz.

(3) What is the number of feet in $\frac{4}{5}$ of a mile; and the number of yards in $\frac{7}{8}$ of a league?

Answers: 4224ft. and 4620yds.

(4) Required the values of $\frac{1}{32}$qrs., $\frac{3}{8}$bush. and $\frac{5}{7}$pks.
Answers: 1pk.; 1pk. 1gal. and 1gal. 1qt. $1\frac{3}{7}$pts.

(5) What is the value of $\frac{2}{15}$ of a month of 28 days, and of $\frac{5}{18}$ of a week?

Answers: 3 days. 17hrs. 36min. and 1 day. 22hrs. 40min.

(6) Required the sum and difference of $\frac{2}{3}$ of 5 guineas and $\frac{3}{4}$ of $\frac{7}{9}$ of a pound.

Answers: £4. 1*s*. 8*d*. and £2. 18*s*. 4*d*.

(7) Add together $\frac{5}{9}$ of a guinea, $\frac{3}{16}$ of a pound, $\frac{7}{10}$ of a crown and $\frac{5}{8}$ of a shilling.

Answer: 19*s*. $6\frac{1}{2}d$.

(8) Find the sum of $\frac{3}{5}$ of 6*s*. 8*d*., $\frac{5}{7}$ of £2. 3*s*. 9*d*. and $\frac{9}{11}$ of £4. 14*s*. 5*d*.: also of $\dfrac{15\frac{3}{5}}{7\frac{1}{5}}$£., $\frac{1}{3}$ of £$140\frac{21}{40}$ and $1\frac{13}{120}$ guineas.

Answers: £5. 12*s*. 6*d*. and £50.

(9) Required the value of $\frac{2}{7}$ of $\dfrac{1}{3\frac{1}{5}}$ of $\frac{1}{14}$ of £$6304\frac{2}{3}$: and find the worth of $\frac{1}{5}$ of $\frac{3}{16}$ of this value.

Answer: £40. 4*s*. 2*d*. and £1. 10*s*. $1\frac{3}{4}d$. $\frac{1}{2}f$.

(10) Find the value of $\dfrac{3}{4}$ of a guinea + $\dfrac{3}{8}$ of a crown + $\dfrac{3}{5}$ of 7*s*. 6*d*. $-\dfrac{3}{4}$ of 2*d*.

Answer: £1. 2*s*.

RULES OF PRACTICE.

99. We shall here shew how the primitive fractions, as defined in Article (69), may be applied to the *practical* calculation of prices, when the price of an unit of any denomination is supposed to be given: and the tediousness of the *enunciations* of the rules at length, will be a sufficient excuse for the mere *indications* of the processes to be employed, by means of examples.

(1) *Simple Practice.*

Ex. 1. Required the value of 1298 at 17*s*. $9\frac{3}{4}$*d*. each, where the unit in 1298 may be of any kind whatever.

Here, we shall have no difficulty in tracing the reason of the following process:

s. d.		£. s. d.	
10 . 0	$\frac{1}{2}$	1298 . 0 . 0	= price at £1.:
5 . 0	$\frac{1}{2}$	649 . 0 . 0	= price at 10*s*.:
2 . 6	$\frac{1}{2}$	324 . 10 . 0	= price at 5*s*.:
0 . 3	$\frac{1}{10}$	162 . 5 . 0	= price at 2*s*. 6.:
0 . $0\frac{3}{4}$	$\frac{1}{4}$	16 . 4 . 6	= price at 3*d*.:
		4 . 1 . $1\frac{1}{2}$	= price at $\frac{3}{4}$*d*.:
		£1156 . 0 . $7\frac{1}{2}$	= price at 17*s*. $9\frac{3}{4}$*d*.:

where it is observed that the denomination of the result is the same as that of the unit assumed, which is here £1: and it is generally most convenient, when possible, to use the *aliquot parts* of the denomination next superior to the highest denomination of the price proposed.

Ex. 2. Find the value of 750 at £5. 8*s*. 4*d*.

Here, proceeding as before, and by Compound Multiplication, we have the following solutions.

By Practice.

s. d.	£.	
6 . 8	$\frac{1}{3}$	750
		5
		3750
1 . 8	$\frac{1}{4}$	250
		62 . 10
		£4062 . 10

By Compound Multiplication.

£. s. d.	
5 . 8 . 4	
10	$10 \times 5 \times 5 \times 3 = 750$
54 . 3 . 4	
5	
270 . 16 . 8	
5	
1354 . 3 . 4	
3	
£4062 . 10 . 0	

and the number of figures in the former being much less than the number in the latter, the *practical* advantage of the method is at once apparent.

(2) *Compound Practice.*

Ex. 1. What is the price of 3cwt. 2qrs. 16lbs. at £3. 7*s*. 8*d*. per cwt.?

Here, the following process will be manifest:

		£.	s.	d.	
2qrs.	$\frac{1}{2}$	3 .	7 .	8	= price of 1cwt:
				3	
		10 .	3 .	0	= price of 3cwt:
14lbs.	$\frac{1}{4}$	1 .	13 .	10	= price of 2qrs. or $\frac{1}{2}$ of 1 cwt:
2lbs.	$\frac{1}{7}$	0 .	8 .	$5\frac{1}{2}$	= price of 14lbs. or $\frac{1}{4}$ of 2qrs:
		0 .	1 .	$2\frac{1}{2}$	= price of 2lbs. or $\frac{1}{7}$ of 14lbs:
		£ 12 .	6 .	6	= price of 3cwt. 2qrs. 16lbs.

Ex. 2. If a servant's wages be £25. 15*s*. for 12 months, how much should be received for 7 months?

Proceeding as before, we have

		£.	s.	d.	
6mo.	$\frac{1}{2}$	25 .	15 .	0	= wages for 12 months:
1mo.	$\frac{1}{6}$	12 .	17 .	6	= wages for 6 months:
		2 .	2 .	11	= wages for 1 month:
		£ 15 .	0 .	5	= wages for 7 months:

and here, as well as in the preceding examples, the operations may be *divested* of the explanations affixed to their right hand, without affecting the clearness of the principles.

Ex. 3. Required the value of $2937\frac{1}{2}$ at $10\frac{3}{4}d$.

d.	s.	s.
6	$\frac{1}{2}$	2937
d. 4	s. $\frac{1}{3}$	1468 . 6
$\frac{1}{2}$	$\frac{1}{8}$	979 . 0
$\frac{1}{4}$	$\frac{1}{2}$	122 . $4\frac{1}{2}$
		61 . $2\frac{1}{4}$
$\frac{1}{2}$ of $10\frac{3}{4}d.$ =		0 . $5\frac{1}{4}$. $\frac{1}{2}f.$
	2,0)	263,1 . 6 . $\frac{1}{2}f.$
		£ 131 . 11*s*. 6*d*. $\frac{1}{2}f.$

where the *former* part of the operation is simple practice, and the *latter* is compound.

Examples for Practice.

(1)	2710 at $1\frac{1}{2}d.$	Ans.: £16. $18s.$ $9d.$
(2)	3467 at $3\frac{3}{4}d.$	Ans.: £54. $3s.$ $5\frac{1}{4}d.$
(3)	659 at $1s.$ $7\frac{3}{4}d.$	Ans.: £54. $4s.$ $7\frac{1}{4}d.$
(4)	1250 at $2s.$ $3\frac{3}{4}d.$	Ans.: £144. $10s.$ $7\frac{1}{2}d.$
(5)	328 at $8s.$ $5\frac{1}{2}d.$	Ans.: £138. $14s.$ $4d.$
(6)	7351 at $14s.$ $9\frac{1}{4}d.$	Ans.: £5429. $0s.$ $4\frac{3}{4}d.$
(7)	7777 at $17s.$ $8\frac{3}{4}d.$	Ans.: £6893. $19s.$ $8\frac{3}{4}d.$
(8)	537 at £1. $7s.$ $2\frac{1}{2}d.$	Ans.: £730. $10s.$ $10\frac{1}{2}d.$
(9)	2937 at £2. $11s.$ $10\frac{3}{4}d.$	Ans.: £7620. $18s.$ $0\frac{3}{4}d.$
(10)	$7432\frac{1}{2}$ at $13s.$ $6\frac{1}{2}d.$	Ans.: £5032. $8s.$ $5\frac{1}{4}d.$
(11)	$1530\frac{1}{4}$ at $15s.$ $9d.$	Ans.: £1205. $1s.$ $5\frac{1}{4}d.$
(12)	$6147\frac{3}{4}$ at $17s.$ $6\frac{1}{2}d.$	Ans.: £5392. $1s.$ $9\frac{1}{4}d.$ $\frac{1}{2}f.$
(13)	$217\frac{1}{4}$ at £2. $17s.$ $7\frac{1}{2}d.$	Ans.: £625. $19s.$ $0\frac{1}{4}d.$ $\frac{1}{2}f.$
(14)	$769\frac{3}{5}$ at £1. $12s.$ $6d.$	Ans.: £1250. $12s.$ $0d.$
(15)	$674\frac{5}{8}$ at £3. $19s.$ $6\frac{1}{2}d.$	Ans.: £2683. $0s.$ $9\frac{1}{2}d.$ $\frac{1}{4}f.$

(16) Required the price of 3cwt. 2qrs. 17lbs., at £1. $5s.$ $8d.$ per quarter.

Answer: £18. $14s.$ $11d.$

(17) Find the cost of 57cwt. 3qrs. 14lbs., at £5. $9s.$ $6d.$ per cwt.

Answer: £316. $17s.$ $3\frac{3}{4}d.$

(18) Find the price of 2cwt. 3qrs. 12lbs., at £1. $7s.$ $6d.$ per cwt.

Answer: £3. $18s.$ $6\frac{3}{4}d.$ $\frac{3}{7}f.$

(19) How much will the carriage of 5 packages, each weighing 4cwt. 3qrs. 21lbs., come to at $12s.$ $6d.$ per ton?

Answer: $15s.$ $5\frac{5}{32}d.$

(20) What is the value of 45oz. 6dwts. 7grs., at $5s.$ $10d.$ an ounce?

Answer: £13. $4s.$ $4\frac{1}{48}d.$

(21) Required the value of 16yds. 2ft. 10in., at $2s.$ $6\frac{1}{2}d.$ a yard.

Answer: £2. $3s.$ $0\frac{3}{4}d.$ $\frac{2}{9}f.$

(22) Find the value of 44ac. 2ro. 25po., at £55. 16*s*. $7\frac{1}{2}$*d*. an acre.

Answer: £2493. 4*s*. $3\frac{1}{4}$*d*. $\frac{11}{16}$*f*.

(23) Determine the price of 5cwt. 3qrs. $11\frac{1}{2}$lbs., at £5. 3*s*. $11\frac{1}{2}$*d*. per cwt.

Answer: £30. 8*s*. $5\frac{97}{448}$*d*.

(24) Find the price of 35qrs. 7bush. $3\frac{1}{4}$pks. of wheat, at 58*s*. 6*d*. per quarter.

Answer: £105. 4*s*. $7\frac{1}{2}$*d*. $\frac{3}{16}$*f*.

(25) If a gallon of mixture cost 5*s*. $3\frac{3}{4}$*d*. $\frac{1}{6}$*f*., it is required to ascertain the price of 53gals. 2qts. $1\frac{1}{7}$pt.

Answer: £14. 5*s*. $2\frac{1}{4}$*d*. $\frac{23}{35}$*f*.

100. MISCELLANEOUS QUESTIONS.

(1) If a person have an eighth of a fifth part of £2000, what is the value of his share?

$$\text{Here, } \frac{1}{8} \text{ of } \frac{1}{5} = \frac{1 \times 1}{8 \times 5} = \frac{1}{40}:$$

$$\text{therefore his share is } \frac{1}{40} \text{ of } \overset{\pounds.}{2000} = \frac{\overset{\pounds.}{2000}}{40} = \overset{\pounds.}{50}:$$

or, taking it in another point of view, we have.

$$\frac{1}{5} \text{ of } \overset{\pounds.}{2000} = \frac{\overset{\pounds.}{2000}}{5} = \overset{\pounds.}{400}:$$

$$\text{whence, } \frac{1}{8} \text{ of } 400 = \frac{400}{8} = \overset{\pounds.}{50}, \text{ as before.}$$

(2) The aggregate of $\frac{2}{3}$ and $\frac{3}{5}$ of a sum of money is £133: what is the sum?

$$\text{Here, } \frac{2}{3} + \frac{3}{5} = \frac{10+9}{15} = \frac{19}{15};$$

$$\text{therefore, } \frac{19}{15} \text{ of the sum is } \pounds 133:$$

$$\text{whence, } \frac{1}{15} \text{ of the sum is } \frac{1}{19} \text{ of } \pounds 133 = \pounds 7:$$

and the sum itself must therefore be $7 \times 15 = \pounds 105$.

(3) Find what quantity multiplied by $\frac{2}{3}$ of $\frac{4}{5}$ of $3\frac{1}{2}$, gives a result equal to $\frac{7}{9}$.

$$\text{First, } \frac{2}{3} \text{ of } \frac{4}{5} \text{ of } 3\tfrac{1}{2} = \frac{2}{3} \times \frac{4}{5} \times \frac{7}{2} = \frac{28}{15}:$$

$$\text{therefore, the required fraction } \times \frac{28}{15} = \frac{7}{9}:$$

but since, when *equal* quantities are multiplied by the *same* quantity, the results are equal, we have

$$\text{the required fraction } \times \frac{28}{15} \times \frac{15}{28} = \frac{7}{9} \times \frac{15}{28} = \frac{5}{12}:$$

that is, the required fraction $= \frac{5}{12}$, because $\frac{28}{15} \times \frac{15}{28} = 1$.

(4) Find what number of times £24. 16*s*. $4\frac{1}{2}$*d*. is contained in £335. 1*s*. $0\frac{3}{4}$*d*.

$$\text{Here, } \overset{\pounds.}{24} \,.\, \overset{s.}{16} \,.\, \overset{d.}{4\tfrac{1}{2}} = \frac{\overset{\pounds.}{23826}}{960}:$$

$$\text{and } 335 \,.\, 1 \,.\, 0\tfrac{3}{4} = \frac{321651}{960}:$$

whence, the required number of times will be

$$= \frac{\overset{\pounds.}{321651}}{960} \div \frac{\overset{\pounds.}{23826}}{960} = \frac{321651}{960} \times \frac{960}{23826}$$

$$= \frac{321651}{23826} = \frac{27}{2} = 13\tfrac{1}{2};$$

that is, £335. 1*s*. $0\frac{3}{4}$*d*. is equal to $13\frac{1}{2}$ times £24. 16*s*. $4\frac{1}{2}$*d*.: and the latter being regarded as a compound *unit* and represented by 1, the former will be represented by $13\frac{1}{2}$.

(5) A person possessed of $\frac{2}{5}$ths of a coal mine, sells $\frac{3}{4}$ths of his share for £2000; what is the whole mine worth?

Here, if the mine be considered the unit and be represented by 1,

$$\text{we have } \frac{3}{4} \text{ of } \frac{2}{5} \text{ of it} = \frac{3}{10}, \text{ which is}$$

the fraction of it sold for £2000: that is, $\frac{3}{10}$ is worth £2000:

therefore $\frac{1}{10}$ is worth $\frac{1}{3}$ of £2000, or £666. 13*s*. 4*d*.: and 1, or the whole mine, will therefore be worth

$$(£666.\ 13s.\ 4d.) \times 10 = £6666.\ 13s.\ 4d.$$

(6) *A* can do a piece of work in 5 days, *B* in 6 and *C* in 7: how much of it can they jointly do in 2 days?

Assuming the piece of work to be represented by the unit or 1, we have

$$\frac{1}{5} = \text{part done by } A \text{ in 1 day,}$$

$$\frac{1}{6} = \ldots\ldots\ldots\ldots\ldots\ldots\ B\ \ldots\ldots\ldots\ldots,$$

$$\frac{1}{7} = \ldots\ldots\ldots\ldots\ldots\ldots\ C\ \ldots\ldots\ldots\ldots,$$

$$\text{therefore } \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{42 + 35 + 20}{210} = \frac{107}{210}$$

is the part done jointly by *A*, *B* and *C* in 1 day: whence, the work done by them jointly in 2 days, will be

$$\frac{107}{210} \times 2 = \frac{214}{210} = 1\tfrac{2}{105};$$

that is, they could finish the whole work in 2 days, and $\frac{2}{105}$ of the same work besides.

Hence also, the time in which they would exactly complete the work is

$$1 \div \frac{107}{210} = 1 \times \frac{210}{107} = \frac{210}{107} = 1\tfrac{103}{107} \text{ days.}$$

(7) One half of the trees in an orchard are apple trees, one-fourth are pear trees, one-sixth plum trees, and there are 50 cherry trees: what number of trees does it contain?

Representing the number of trees in the orchard by the unit or 1, we have

$\frac{1}{2}$ = number of apple trees:

$\frac{1}{4}$ = number of pear trees:

$\frac{1}{6}$ = number of plum trees:

and the sum of these numbers $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$

whence, the number of cherry trees $= 1 - \frac{11}{12} = \frac{1}{12}$:

that is, $\frac{1}{12}$ of the whole number of trees $= 50$;

and the whole number is therefore $= 50 \times 12 = 600$.

This solution is easily verified; for, we have

the number of apple trees $= \frac{1}{2}$ of $600 = 300$:
the number of pear trees $= \frac{1}{4}$ of $600 = 150$:
the number of plum trees $= \frac{1}{6}$ of $600 = 100$:
the number of cherry trees .... $= 50$:

and the number of trees in the orchard $= 600$.

Examples for Practice.

(1) If $\frac{3}{16}$ of a lottery ticket cost £4. 10*s.*, what is the price of $\frac{1}{6}$ of a ticket?

Answer: £4. 16*s.*

(2) The owner of $\frac{4}{17}$ of a ship sold $\frac{3}{11}$ of $\frac{2}{9}$ of his share for £$12\frac{4}{33}$; what would $\frac{2\frac{1}{2}}{4\frac{1}{4}}$ of $\frac{2}{5}$ of it cost, at the same rate?

Answer: £200.

(3) Express a degree of $69\frac{1}{2}$ miles in metres, where 32 metres are equal to 35 yards.

Answer: $111835\frac{3}{7}$ metres.

(4) If I import 5763 bushels of wheat for £1800. 18*s.* 9*d.*, and pay an import duty of $10\frac{1}{3}$ per cent. on the money thus laid out, what is the duty per bushel?

Answer: $7\frac{3}{4}$*d.*

(5) Find the value of the metre of France, in terms of the foot of Cremona, if 48 Cremonese feet = 56 English feet, and if the metre be $39\frac{371}{1000}$ English inches.

Answer: $2\frac{11371}{14000}$ feet.

(6) What number is that, whereof the part expressed by $\frac{1}{3} + \frac{1}{4} + \frac{1}{6}$ is 45?

Answer: 60.

(7) If a person lay out $\frac{5}{9}$ of his income in board and lodging, $\frac{1}{6}$ in clothes and save £60. a year: what is his income?

Answer: £216.

(8) After detaching $\frac{11}{28}$ and $\frac{12}{35}$ of a company of soldiers, the general had 1110 left: required his original force.

Answer: 4200 soldiers.

(9) What is the capacity of a vessel, out of which when a third of it is empty, 35 gallons being drawn, there remains $\frac{3}{8}$ of the whole content?

Answer: 120 gallons.

(10) A post has one-fourth of its length in the mud, one-third in the water and 10 feet above the water: find its whole length.

Answer: 24 feet.

(11) If $\frac{11}{20}$ of an estate be left to the elder and the remainder to the younger of two children, and the difference of their legacies be £225: find the value of the estate.

Answer. £2250.

(12) A ship and its cargo are worth £17600. and the cargo is worth seven times as much as the ship: find the values of the ship and cargo.

Answer: £2200. and £15400.

(13) Of a field $\frac{1}{5}$ is meadow, $\frac{3}{8}$ is arable and the remainder is 1ac. 3ro. 26po: find the quantities of meadow and arable land.

Answer: 3ro. 24po. and 1ac. 2ro. 30po.

(14) *A* met two beggars *B* and *C*, and having $\dfrac{3\frac{7}{11}}{4\frac{2}{7}}$ of $\dfrac{10\frac{5}{7}}{7\frac{1}{2}}$ of $\frac{77}{540}$ of a moidore of 27*s*. in his pocket, gave $\frac{1}{7}$ of $\frac{3}{4}$ of it to *B* and $\frac{3}{5}$ of the remainder to *C*: what did each receive?

Answer: *B* received 6*d*. and *C* had 2*s*. 6*d*.

(15) *A* had at first £1. 8*s.*; and *B*, when he had paid $2\frac{3\frac{1}{7}}{1\frac{2}{9}}$ of £1. 11*s.* 6*d.* to *A*, found that he had remaining $\frac{1}{43}$ of what *A* then had: what had *B* at first?

Answer: £7. 8*s.*

(16) *A* can reap a field in 5 days and *B* in 6 days, working 11 hours a day; find in what time *A* and *B* can reap it together, working 10 hours a day.

Answer: 3 days.

(17) If *A* and *B* can do a piece of work in 18 days, *A* and *C* in 12 days and *B* and *C* in 9 days, find the time in which *A*, *B* and *C* can do it.

Answer: 8 days.

(18) If a cask be emptied by two taps in 4 and 6 hours respectively, in what time will it be emptied by both of them together, the rates of efflux remaining the same throughout?

Answer: 2hrs. 24min.

(19) *A*, *B* and *C* can perform a piece of work in 12 hours: also, *A* and *B* can do it in 16 hours and *A* and *C* in 18 hours: what part of the work can *B* and *C* do in $9\frac{1}{7}$ hours?

Answer: $\frac{4}{9}$.

(20) A cistern is filled by two spouts in 20 and 24 minutes respectively and emptied by a tap in 30 minutes: what portion of it will be filled in 15 minutes when they are all left open together, the influx and efflux being uniform?

Answer: $\frac{7}{8}$.

(21) In an orchard, $\frac{1}{3}$ of the trees are apple trees, $\frac{1}{4}$ pear trees, $\frac{1}{5}$ cherry trees, $\frac{1}{6}$ filbert trees and there are 12 walnut trees: what is the number of each sort?

Answer: 80 apple trees, 60 pear trees, 48 cherry trees, 40 filbert trees and 12 walnut trees.

(22) A person after paying away one third of his money together with £10., finds that he has remaining £15. more than its half: what money had he?

Answer: £150.

(23) A farmer pays a corn-rent of 5 quarters of wheat and 3 quarters of barley, Winchester measure: what is his rent, wheat being at 60*s*. and barley at 54*s*. per quarter, imperial measure, it being assumed that 32 imperial gallons are equivalent to 33 Winchester gallons?

Answer: £22. 8*s*.

(24) *A* can do a piece of work in 4 hours, *B* can do $\frac{3}{4}$ of the remainder in 1 hour and *C* can then finish it in 20 minutes: in what time can *A*, *B*, *C* together do it?

Answer: $1\frac{1}{2}$ hours.

(25) A person leaves $\frac{1}{3}$ of £10000. to his wife, $\frac{1}{2}$ to his son and the rest to his daughter. The wife leaves $\frac{2}{5}$ of her legacy to the son and the rest to the daughter: but the son adds his fortune to his sister's and gives her $\frac{1}{3}$ of the whole. How much does the sister gain by this proceeding?

Answer: £333. 6*s*. 8*d*.

(26) If *A* in 2 days can do as much work as *C* in 3 days and *B* in 5 days as much as *C* in 4 days: what time will *B* require to execute a piece of work which *A* can accomplish in 6 weeks?

Answer: $11\frac{1}{4}$ weeks.

(27) A cistern has three pipes *A*, *B*, *C*: *A* and *B* can fill it in 3 and 4 hours respectively and *C* can empty it in 1 hour: if these pipes be opened in order at 1, 2 and 3 o'clock, prove that the cistern will be empty at 12 minutes past 5.

(28) If *A* can do half a piece of work in 3 hours, which is twice as much as *B* can do, and *A*, *B*, *C* can together do the whole in $2\frac{1}{2}$ hours: shew that *C* can do in 5 hours as much as *B* can do in 9 hours.

CHAPTER V.

THE THEORY OF DECIMALS.

101. Def. In the Notation of Integers, it has been seen that the figures in the units' place alone retain their *absolute* values, whilst the *local* values of figures in other situations increase *tenfold* for every figure we advance towards the left hand from that place. Therefore, in beginning at the *left*-hand figure of any number and proceeding towards the *right* hand, it follows that the *local* value of every figure will be a *tenth* part of that which immediately precedes it: and if we suppose figures to be situated to the right of the units' place, and this kind of tenfold *subdivision* to be extended to them, it is manifest that the local values of such figures in order from the place of units, will be a *tenth*, a *hundredth*, a *thousandth*, &c., parts of their absolute values.

Hence, we are enabled to represent integers and fractions by one uniform system of notation, by merely marking the *place of units*: and whilst *Integers* are expressed by figures in the units' place and in places to the *left* of it, *Fractions* will be represented by figures situated in places on the *right* of the units, called the places of *tenths*, *hundredths*, *thousandths*, &c.

In this manner originates the system of *Decimals*, being merely an extension of the Notation of Integers: and though there are decimals of all denominations as *Decimal Integers*, yet from the circumstance of the system representing only *tenth*, *hundredth*, *thousandth*, &c., parts of the unit, all *fractions* belonging to it are termed *Decimal Fractions*, in contradistinction to *Vulgar Fractions*, whereof the denominations may be any parts whatever.

Whence, *Decimals* may be *defined* to be *Fractions* whose denominators are 10, 100, 1000, &c., these denominators not being *written* as in Vulgar Fractions, but *expressed* by the position of what is called the *Decimal Point*.

NOTATION AND NUMERATION OF DECIMALS.

102. If we suppose the digit 1 to occupy the units' place, the following scheme will point out the denominations of the figures to the left and right of it, and it may be extended so as to include both integers and fractions of all local values whatever:

&c.	Thousands.	Hundreds.	Tens.	Units.	Tenths.	Hundredths.	Thousandths.	&c.
&c.	4	3	2	1	2	3	4	&c.

and a mixed quantity, formed of integers and fractions, is separated into its *integral* and *fractional* portions by means of the *Decimal Point* placed on the right of the units' place, which dispenses with the description of the local denominations, above given.

Thus, in 4321 .2345, the figures 4321 on the *left* of the point denote so many integers, and the figures 2345 on the *right* of it, so many fractions, namely, 2 *tenths*, 3 *hundredths*, 4 *thousandths*, and so on: and the *expressing* and *reading* of *Decimals* will evidently be conducted upon the respective principles of the *Notation* and *Numeration* of integers: also, inasmuch as *Integers* denote assemblages of two or more *units*, *Decimals* will represent assemblages of two or more tenth, hundredth, &c., *parts* of an unit.

Relation of Decimals to Vulgar Fractions.

103. From the statements made in the preceding Articles, it is obvious that every magnitude made up of one or more decimals is equivalent to, and may be expressed by, one or more vulgar fractions having 10, 100, 1000, &c., for their denominators: and that all *mixed* quantities expressed decimally may be represented by means of *whole* numbers and *vulgar fractions* of similar denominations.

104. *Every Decimal may be expressed exactly by a vulgar fraction.*

$$\text{Thus, } .327 = \frac{3}{10} + \frac{2}{100} + \frac{7}{1000} = \frac{327}{1000}:$$

$$.0459 = \frac{0}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{9}{10000} = \frac{459}{10000}:$$

$$13.816 = 13 + \frac{8}{10} + \frac{1}{100} + \frac{6}{1000} = \frac{13816}{1000}:$$

and hence we infer that a decimal will always be equivalent to the vulgar fraction formed by taking it considered *integral,* for the numerator, and by placing 1, with as *many* ciphers as there are *decimal places* contained in it, for the denominator.

In these instances, we see that the reduction to a common denominator, so tedious in vulgar fractions, is entirely dispensed with, and the *immediate* comparison of fractional quantities is one of the great advantages of the system.

Since $.327 = \frac{327}{1000}$, it is evident that .327 may be read *327 thousandths,* the decimal having always the denomination of its *last* figure on the right hand: and conversely, every vulgar fraction having 10, 100, 1000, &c., for its denominator, may be immediately represented by a decimal, by beginning at the figure on the *right* hand of the numerator, and pointing off as many decimal places, supplied with ciphers toward the *left,* if necessary, as there are ciphers in the denominator.

105. *Ciphers annexed to the right hand of a Decimal Fraction have no effect upon its value.*

$$\text{Thus, } .37 = \frac{37}{100}, \quad .370 = \frac{370}{1000} = \frac{37}{100},$$

$$.3700 = \frac{3700}{10000} = \frac{37}{100}, \text{ and so on:}$$

as appears also from the consideration, that there are *no* thousandths, &c., in addition to the tenths and hundredths expressed by .37.

106. *Every cipher affixed to the left hand of a Decimal Fraction after the point diminishes its value tenfold.*

Thus, $.43 = \frac{43}{100}$, $.043 = \frac{43}{1000}$, $.0043 = \frac{43}{10000}$, &c.; where each fraction is a tenth part of that which immediately precedes it: and indeed this is evident from the circumstance of every figure being reduced *one* denomination lower by means of *each* cipher.

Hence, *Multiplication* and *Division* by 10, 100, 1000, &c., are immediately effected, by shifting the decimal point *one, two, three,* &c., places towards the *right* and *left* respectively.

107. *Every Vulgar Fraction may be expressed either accurately or approximately by a Decimal.*

Let us take the fractional quantities $\frac{3}{8}$ and $4\frac{7}{125}$: then,

$$\frac{3}{8} = \frac{3000}{8000} = \frac{\frac{1}{8}(3000)}{1000} = \frac{375}{1000} = .375:$$

$$\text{and } 4\tfrac{7}{125} = 4 + \frac{7000}{125000} = 4 + \frac{\frac{1}{125}(7000)}{1000}$$

$$= 4 + \frac{56}{1000} = 4 + .056 = 4.056:$$

whence we have the following Rule.

Rule. Divide the numerator of the fraction with as many ciphers annexed to the right of it, as may be deemed necessary, by the denominator: and the quotient comprising as *many* decimal places as there are *ciphers* annexed, will be the decimal required.

If the division do not terminate, neither is the corresponding decimal *finite*, and the vulgar fraction is expressed only *approximately* by the decimal fraction thus found: three or four ciphers are generally sufficient for all practical purposes, but the approximation will be nearer, the further the division is continued, inasmuch as by *every succeeding* step of the operation, a decimal fraction of an *inferior* denomination is added to the value already obtained.

Examples for Practice.

(1) Transform the decimals, .5, .75, .625, .1875, 2.56, to vulgar fractions in their lowest terms.

Answers: $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{3}{16}, \frac{64}{25}$.

(2) Find the simplest vulgar fractions equivalent to the decimals: .432, .00625, .1015625, .0109375.

Answers: $\frac{54}{125}$, $\frac{1}{160}$, $\frac{13}{128}$, $\frac{7}{640}$.

(3) What vulgar fractions are equivalent to the mixed decimals:

1.075, 3.01875, 7.0046875, 13.0005859375?

Answers: $\frac{43}{40}$, $\frac{483}{160}$, $\frac{4483}{640}$, $\frac{66563}{5120}$.

(4) Convert the following vulgar fractions:

$\frac{3}{4}$, $\frac{5}{8}$, $\frac{399}{128}$, $\frac{13}{1600}$, $\frac{2504}{625}$ into decimals.

Answers: .75, .625, 3.1171875, .008125, 4.0064.

(5) What are the decimal fractions equivalent to the vulgar fractions $\frac{4}{25}$, $\frac{9}{125}$, $\frac{17}{2560}$, $\frac{238}{6250}$?

Answers: .16, .072, .006640625, .03808.

(6) Express $3\frac{4}{25}$, $4\frac{5}{16}$, $1\frac{2}{15}$ of $11\frac{1}{4}$, $\frac{12\frac{3}{4}}{10\frac{1}{5}}$, as decimals.

Answers: 3.16, 4.3125, 12.75, 1.25.

(7) Represent the approximate values of

$\frac{1}{3}$, $\frac{2}{7}$, $\frac{3}{11}$ and $\frac{5}{13}$,

by three or more places of decimals.

Answers: .333 &c., .2857142 &c., .2727 &c., .384615 &c.

I. ADDITION OF DECIMALS.

108. Rule. Place the quantities so that all the decimal points may be in the same vertical line, to insure the combination of those of the *same* denominations: and add them together as in integers, taking care to place the decimal point in the sum, immediately under those of the quantities proposed.

Examples.

$$\begin{array}{r} .419 \\ .0256 \\ .08 \\ .21734 \\ \hline .74194 \end{array} \qquad \begin{array}{r} 25.6 \\ 4.805 \\ .009 \\ 653.27 \\ \hline 683.684 \end{array}$$

Proof by Vulgar Fractions.

Using only the latter example, we have

$$25.6 = \frac{256}{10} = \frac{25600}{1000}:$$

$$4.805 = \frac{4805}{1000}.$$

$$.009 = \frac{9}{1000}:$$

$$653.27 = \frac{65327}{100} = \frac{653270}{1000}:$$

whence, the sum =

$$\frac{25600 + 4805 + 9 + 653270}{1000} = \frac{683684}{1000} = 683.684, \text{ as above.}$$

Hence, decimals are said to be reduced to a *common denominator,* when ciphers are supplied so that there is the *same* number of decimal places in each.

II. SUBTRACTION OF DECIMALS.

109. RULE. Place the less quantity under the greater as in Addition; suppose ciphers to be supplied if necessary, in the upper line; and the difference, found as in integers, will have as many decimal places as are contained in each, either expressed or understood.

Examples.

$$\begin{array}{r} .7053 \\ .6729 \\ \hline .0324 \end{array} \qquad \begin{array}{r} 41.62 \\ 34.917 \\ \hline 6.703 \end{array}$$

Proof by Vulgar Fractions.

In the latter of these examples, we have

$$41.62 - 34.917 = \frac{4162}{100} - \frac{34917}{1000}$$

$$= \frac{41620 - 34917}{1000} = \frac{6703}{1000} = 6.703,$$

as before; and the necessity of *supposing* the cipher to be supplied is here shewn.

III. MULTIPLICATION OF DECIMALS.

110. RULE. Multiply together the quantities proposed as if they were integers: and the product will contain as many places of decimals, as there are decimal places in the multiplicand and multiplier together.

Examples.

$$\begin{array}{r} .45 \\ .21 \\ \hline 45 \\ 90 \\ \hline .0945 \end{array} \qquad \begin{array}{r} 6.27 \\ 15.9 \\ \hline 5643 \\ 3135 \\ 627 \\ \hline 99.693 \end{array}$$

where the former product, found as if all the quantities were *whole numbers* would manifestly be *ten thousand* times too great, because 45 and 21 are a hundred times as great as .45 and .21 respectively; and therefore the true product is obtained by placing the decimal point *four* places towards the *left* hand, by Article (106).

Proof by Vulgar Fractions.

The latter product in these examples is

$$6.27 \times 15.9 = \frac{627}{100} \times \frac{159}{10} = \frac{99693}{1000} = 99.693, \text{ as above.}$$

IV. DIVISION OF DECIMALS.

111. RULE. Supply the dividend with ciphers to the right hand, if necessary, and divide exactly as in integers: then the quotient will have a number of decimal places equal to the excess of the number of such places in the dividend above that in the divisor.

Examples.

$$.012)\overline{.241728}\quad 20.144 \qquad\qquad 2.5).1875(.075$$

```
.012).241728      |   2.5).1875(.075
      ------      |        175
      20.144      |        ---
                  |         125
                  |         125
```

In these instances, the numbers of decimal places in the quotients are the excesses of the numbers of decimal places in the dividends above those in the divisors, because the divisors and quotients must together comprise as many places as the dividends, by the last Rule.

Proof by Vulgar Fractions.

Here, $.1875 \div 2.5 = \frac{1875}{10000} \div \frac{25}{10} = \frac{1875}{10000} \times \frac{10}{25} = \frac{75}{1000}$ $= .075$, as before.

If the divisor and dividend have the *same* number of decimal places, the quotient will comprise *no* decimal places as there is no excess: but if there be *more* places in the divisor than in the dividend, ciphers must be supplied so as to render the number in the dividend *not less* than that in the divisor, before the rule can be applied: and the reason of this will be seen in the following example:

$$62.5 \div .025 = \frac{625}{10} \div \frac{25}{1000} = \frac{625}{10} \times \frac{1000}{25} = \frac{625}{25} \times \frac{1000}{10} = 25$$

$\times\, 100 = 2500$, where there is annexed to the right of the quotient obtained as in integers, a number of ciphers equal to the excess of the number of decimal places in the divisor above that in the dividend, the quotient being the integral quantity 2500.

If the division do not terminate, *three* or *four* decimal places in the quotient are generally sufficient.

Examples for Practice.

(1) Express in decimals; One and Fifty four hundredths; Twenty four and Seventy nine thousandths; Three hundred and fifteen, Eight thousandths and Fifty millionths.

(2) Find the sum of .295, 3.086, 12.87, .0051, 729.54: also, of 3608.26, 360.826, 36.0826, 3.60826, .22314.

(3) Add together 36.053, .0079, .000952, 417, 85.5803, .0000501: and prove it by vulgar fractions.

(4) Find the difference of 27.903 and .054: of 7295.06 and 254.738: of 35.08989 and 3.508989.

(5) Required the excess of 2.057 above 1.0097 and of 3.025 above .003025: and prove the results.

(6) Required the product of .718 and .57: of 16.8 and .0024: of 144 and .0625: of 12.5 and .062216.

(7) Multiply 270.56 by .37025: .00579 by 3796.8: .384759375 by .00032: and prove them.

(8) Find the continued product of $.\dot{2}75$, 2.75 and 27.5: also, of 3.24, $2\frac{15}{16}$ and .0028.

(9) Required the quotient of 35.9424 by 7.02: of .278831 by .653: of 11.444495 by 4.735: of .020872522 by .08635: and prove them by vulgar fractions.

(10) Divide .0257 by .0041: 325.46 by .0187: .0719 by 27.53: to three or more places of decimals.

(11) Find the quotient of 1.68 by .024: of 971.7 by .123: of 142.025 by .0437: of 84.375 by .00375: and prove the results by vulgar fractions.

(12) Simplify the arithmetical expressions, $5-3.22+2.333-1.4444$ and $75.012-7.50123+.7501234-.075012345$.

Answers: 2.6686 and 68.185881055.

(13) Express in the decimal notation, the value of

$$8.0625-6\tfrac{1}{25}-.00375+1.09236-\frac{25679}{10000}.$$

Answer: .54321.

(14) Simplify 1.26 of $66\frac{2}{3} + 5\frac{1}{6}$ of 1.0375 and $3\frac{5}{8}$ of $.003 - .0011$ of $7\frac{1}{4}$.

Answers: 89.395 and .0029.

(15) Reduce to decimal forms, the following expressions:

$$\frac{2.004}{.167} \times \frac{3.375}{4}; \quad \frac{.0295}{3.04} \div \frac{1.18}{.00152} \text{ and}$$

$$3.25 - 2.765 + 3.125 \times 8 - .607095 \div .027.$$

Answers: 10.125, .0000125 and 3.

REDUCTION OF DECIMALS.

112. A general view having now been taken of decimals, we proceed to shew how they may be made to change their denominations when they are considered as belonging to a particular unit; and in what ways they may be adapted to the practical computations in which they are most frequently employed.

113. *A Decimal may be changed into another, whose denomination shall have a given relation to its own.*

Rule. Multiply or divide the decimal by the number or numbers which connect the denominations in order, according as the denomination of the required decimal is lower or higher than its own.

For, from what has been said in the reduction of compound quantities, it is evident that

$$\overset{\text{cwt.}}{.16} = \overset{\text{qrs.}}{.16 \times 4} = \overset{\text{qrs.}}{.64} \text{ and } \overset{\text{lbs.}}{14} = \overset{\text{qrs.}}{\frac{14}{28}} = \overset{\text{qrs.}}{.5} = \overset{\text{cwt.}}{\frac{.5}{4}} = \overset{\text{cwt.}}{.125}.$$

114. *A compound quantity may be exhibited in the form of a Decimal whose denomination is given.*

Rule. Divide the lowest denomination by the number which connects it with the next, and to the left of the quotient affix the number of this denomination: and continue the process till the required denomination is obtained.

Let us take 7fur. 25 po., and express it as the decimal of a mile: then,

$$\overset{\text{po.}}{25} = \overset{\text{fur.}}{\frac{25}{40}} = \overset{\text{fur.}}{.625} = \overset{\text{mi.}}{\frac{.625}{8}} = \overset{\text{mi.}}{.078125} \text{ and } \overset{\text{fur.}}{7} = \overset{\text{mi.}}{\frac{7}{8}} = \overset{\text{mi.}}{.875}:$$

whence, the decimal will be $.078125 + .875 = .953125$ of a mile: also, the same processes are comprised in the following more convenient and *practical form:*

$$\begin{array}{r} 40\,)\,25.000 \\ \hline 8\,)\,7.625000 \\ \hline .953125 \\ \hline \end{array}$$

which suggests the Rule.

115. *The value of a Decimal may be expressed by means of the known parts of its unit.*

RULE. Multiply the decimal by the numbers which connect the successive denominations in order; and the integral parts of the products *taken out*, as they occur, will be the value required.

For, to find the value of .655 of a day, we have

$$\overset{\text{days.}}{.655} = \overset{\text{hrs.}}{.655 \times 24} = \overset{\text{hrs.}}{15.72}:$$

$$\overset{\text{hrs.}}{.72} = \overset{\text{min.}}{.72 \times 60} = \overset{\text{min.}}{43.2}:$$

$$\overset{\text{min.}}{.2} = \overset{\text{sec.}}{.2 \times 60} = \overset{\text{sec.}}{12}:$$

that is, 15hrs. 43min. 12sec is the value required: and the following *form* amounts to the same thing, furnishing the Rule.

$$\begin{array}{rr} & \overset{\text{days.}}{.655} \\ & 24 \\ \hline & 2620 \\ & 1310 \\ \hline \text{hrs.} & 15.720 \\ & 60 \\ \hline \text{min.} & 43.200 \\ & 60 \\ \hline \text{sec.} & 12.000 \end{array}$$

Examples for Practice.

(1) Express £.00375 as decimals of a shilling and a penny.

Answers: .075*s*. and .9*d*.

(2) What decimals of a pound are 8.4 of a penny and .4068 of a farthing?

Answers: £.035 and £.00042375.

(3) Reduce 2.15lbs. to the decimal of a cwt. and 24 yards to the decimal of a mile.

Answers: .01919 &c. and .0136 &c.

(4) Reduce 7oz. 4dwts. to the decimal of 1lb. and 2qrs. $3\frac{1}{9}$nls. to the decimal of an English ell of five quarters.

Answers: .6 and .555 &c.

(5) Reduce 12hrs. 55min. $23\frac{1}{13}$sec. to the decimal of a day and 5days. 12hrs. 25min. 37.92sec. to the decimal of a week.

Answers: .538461 &c. and .788257 &c.

(6) Express 12*s*. $6\frac{3}{4}$*d*. and 15*s*. $9\frac{3}{4}$*d*. as decimals of £1. and £4. 13*s*. $4\frac{1}{2}$*d*. as a decimal of £5.

Answers: .628125, .790625 and .93375.

(7) Reduce 1*s*. 3*d*. to the decimal of 10*s*., 5*s*. to the decimal of 13*s*. 4*d*. and 13*s*. $6\frac{3}{4}$*d*. to the decimal of 15*s*. 6*d*.

Answers: .125, .375 and .875.

(8) Find the values of .45 of £1., .16875 of £3. and 2.36875 of £6.

Answers: 9*s*., 10*s*. $1\frac{1}{2}$*d*. and £14. 4*s*. 3*d*.

(9) Required the values of £.5675, .375cwt., .6875 yds. and 13.3375 acres.

Answers: 11*s*. $4\frac{1}{5}$*d*., 1qr. 14lbs., 2qrs. 3na. and 13ac. 1ro. 14po.

(10) What are the values of .203125qrs. and .73625bush. of corn?

Answers: 1bush. 2pks. 1gal. and 2pks. 1gal. $3\frac{14}{25}$qts.

(11) What is the value of .07 of £2. 10*s.* and of .0474609375 of £10. 13*s.* 4*d.*?

Answers: 3*s.* 6*d.* and 10*s.* $1\frac{1}{2}d.$

(12) Find the value of .5 shillings + .7 crowns + .125 pounds.

Answer: 6*s.* 6*d.*

(13) Reduce £24. 16*s.* $4\frac{1}{2}d.$ and £167. 10*s.* $6\frac{1}{4}d.$ $\frac{1}{2}f.$, to decimals of the same denomination, so as to find how often the former is contained in the latter.

Answer: 6.75.

(14) Express .375 of a guinea + $\frac{3}{16}$ of a crown + .3 of 7*s.* 6*d.* − $\frac{3}{8}$ of 2*d.* as the decimal of 16*s.*

Answer: .6875.

RECURRING DECIMALS.

116. DEF. In the conversion of a vulgar fraction into a decimal, if the division performed according to the rule laid down in Article (107) do not terminate, but the figures of the quotient continually recur in some certain order, the result is called a *recurring* or *circulating* decimal: the quantity repeated is styled its *period* and is termed a *simple* or a *compound repetend* according as it consists of *one* or *more* figures: and the *extent* of the period is denoted by means of *dots* placed over the *first* and *last* of the figures which compose it.

If the quotient comprise other figures besides those which are repeated, it is called a *mixed* circulating decimal, as it consists of a *non-recurring* and a *recurring* part.

Ex. 1. Convert $\frac{1}{3}$ and $\frac{4}{27}$ into decimals.

Proceeding according to the Rule, we have

$$3\underline{)1\,.\,0\,0\,0\,0\,\&c.}\qquad\qquad 27\begin{cases}3\underline{)4\,.\,0\,0\,0\,0\,0\,0\,\&c.}\\ 9\underline{)1\,.\,3\,3\,3\,3\,3\,3\,\&c.}\end{cases}$$
$$.\,3\,3\,3\,3\,\&c.\qquad\qquad\qquad .\,1\,4\,8\,1\,4\,8\,\&c.$$

whence, $\frac{1}{3} = .3333\&c.$ and $\frac{4}{27} = .148148\&c.$:

the former having the *simple* repetend 3 and the latter the *compound* repetend 148: and these *repetends* being denoted by $\dot{3}$ and $\dot{1}4\dot{8}$ respectively,

$$\text{we have } \frac{1}{3} = .\dot{3} \text{ and } \frac{4}{27} = .\dot{1}4\dot{8},$$

which are termed *pure circulates.*

Ex. 2. What is the decimal corresponding to $\frac{5}{36}$? As in the preceding instances, we have

$$36 \begin{cases} 6)\ 5\,.\,0\,0\,0\,0\,0 \ \&c. \\ \overline{6)\ .\,8\,3\,3\,3\,3\ \&c.} \end{cases}$$
$$\overline{.\,1\,3\,8\,8\,8\ \&c.}$$

whence, $\frac{5}{36}$ is equivalent to the *mixed* circulating decimal .13888&c., the *non-recurring* part being 13 and the *recurring* part 8, and it is written $\frac{5}{36} = .13\dot{8}$.

Conversely, every pure or mixed circulating decimal must be equal to, and expressible by, a vulgar fraction.

117. *To find the vulgar fraction which shall be equivalent to a pure recurring decimal.*

Let the circulates be .666 &c. and .9696 &c., or $.\dot{6}$ and $.\dot{9}\dot{6}$: then if, for the sake of conciseness, we suppose the symbols x and y to represent their values, we shall have the following results from Article (106):

$$x = .666\ \&c. \qquad\qquad y = .9696\ \&c.$$
$$10 \textit{ times } x = 6.666\ \&c. \qquad\qquad 100 \textit{ times } y = 96.9696\ \&c.$$

whence, subtracting in each case, the former from the latter, we obtain

$$9 \text{ times } x = 6, \qquad\qquad 99 \text{ times } y = 96,$$

$$\text{and } x = \frac{6}{9} = \frac{2}{3}: \qquad\qquad \text{and } y = \frac{96}{99} = \frac{32}{33}:$$

that is, the vulgar fractions are $\frac{2}{3}$ and $\frac{32}{33}$.

These results may easily be verified, and from them we derive the following Rule.

RULE. Make the repetend the *numerator* of a fraction whose *denominator* shall consist of as many *nines* as there are figures in the said repetend: and this reduced to its simplest terms will be the vulgar fraction required.

118. *To find the vulgar fraction which shall represent the value of a mixed recurring decimal.*

Ex. To ascertain the vulgar fractions equivalent to $.2\dot{7}$ and $.24\dot{5}\dot{7}$, we have, by *abbreviating* the forms,

$$\begin{aligned} x &= .27 & \qquad y &= .24\dot{5}\dot{7} \\ 10x &= 2.\dot{7} & \qquad 100y &= 24.\dot{5}\dot{7} \\ 100x &= 27.\dot{7} & \qquad 10000y &= 2457.\dot{5}\dot{7} \end{aligned}$$

whence, subtracting the second line from the third in each case, we find

$$90x = 25, \qquad\qquad 9900y = 2433,$$

$$\text{and } x = \frac{25}{90} = \frac{5}{18}: \qquad\qquad \text{and } y = \frac{2433}{9900} = \frac{811}{3300}:$$

and these put in the following *forms*,

$$x = \frac{25}{90} = \frac{27-2}{90} \quad\Bigg|\quad y = \frac{2433}{9900} = \frac{2457-24}{9900}$$

furnish us with the following Rule.

RULE. Make the non-recurring and the recurring parts taken *together*, diminished by the non-recurring part taken *alone*, the numerator of a fraction whose denominator shall consist of as many *nines* as there are recurring figures, followed by as many *ciphers* as there are non-recurring figures; and this reduced to its lowest terms will be the vulgar fraction required.

119. It will hence appear that the arithmetical operations upon recurring decimals, may be *correctly* effected by means of the same operations performed upon their equivalent vulgar fractions.

Ex. Let it be required to find the sum, difference, product and quotient, of the recurring decimals $.\dot{6}$ and $.\dot{2}9\dot{6}$.

Here, by the rules, we have $.\dot{6} = \frac{2}{3}$ and $.\dot{2}9\dot{6} = \frac{8}{27}$:

$$\text{therefore, the sum} = \frac{2}{3} + \frac{8}{27} = \frac{26}{27} = .\dot{9}6\dot{2}.$$

$$\text{the difference} = \frac{2}{3} - \frac{8}{27} = \frac{10}{27} = .\dot{3}7\dot{0}:$$

$$\text{the product} = \frac{2}{3} \times \frac{8}{27} = \frac{16}{81} = .\dot{1}9753086\dot{4}:$$

$$\text{the quotient} = \frac{2}{3} \div \frac{8}{27} = \frac{9}{4} = 2.25:$$

the first three of which are recurring decimals and the last a terminating quantity when expressed decimally: and it may be remarked that the same results could have been found from the *immediate* operations, only by means of a less satisfactory process.

120. In the same manner recurring decimals of *specified* units may be treated, and their exact values thence obtained.

Ex. Find the value of $.1\dot{6}$ of a pound sterling.

$$\begin{array}{l} \\ \text{Here,} \end{array} \begin{array}{c} \pounds. \\ .1\dot{6} \end{array} \begin{array}{c} \\ = \end{array} \begin{array}{c} \pounds. \\ \frac{1}{6} \end{array} \begin{array}{c} \\ = \end{array} \begin{array}{c} s. \\ \frac{1}{6} \times \frac{20}{1} \end{array} \begin{array}{c} \\ = \end{array} \begin{array}{c} s. \\ \frac{20}{6} \end{array} \begin{array}{c} \\ = \end{array} \begin{array}{c} s. \quad d. \\ 3 \;.\; 4. \end{array}$$

121. Since, in converting a vulgar fraction into a decimal, either 1, 10, 100, 1000, &c., or their *multiples* are divided by the denominator, it is evident that the decimal will *terminate* or *not*, according as these numbers *are* divisible by the denominator or *not*: whence, as the only incomposite factors of 10, 100, 1000, &c., are 2 and 5, it follows that vulgar fractions whose denominators can be resolved into these factors are *finite* decimals, whilst all others are *recurring* quantities.

Thus, $\frac{3}{50} = \frac{3}{2 \times 5 \times 5} = .06$ a finite decimal:

$\frac{5}{12} = \frac{5}{2 \times 2 \times 3} = .41\dot{6}$ a recurring decimal.

It is clear that at every step of the division, the remainder must be less than the denominator of the fraction,

and therefore the number of *different* remainders can in no case exceed the denominator when diminished by 1: hence, the same operation will have to be *repeated* upon the same figures, whenever the division does not terminate, so that the figures in the remainders and quotients must necessarily *recur* with the same limitation.

Thus, in $\frac{3}{7} = .\dot{4}2857\dot{1}$, in $\frac{5}{13} = .\dot{3}8461\dot{5}$ and in $\frac{25}{14} = 1.7\dot{8}5714\dot{2}$, the periods consist of *six* figures.

Also, to shorten the operation for determining the period when the denominator is not a small number, as for instance in $\frac{5}{17}$, we may proceed as follows:

$$\text{since } \frac{1}{17} = .05882\tfrac{6}{17}, \text{ by actual division:}$$

$$\text{therefore } \frac{6}{17} = .05882\tfrac{6}{17} \times 6 = .35294\tfrac{2}{17}:$$

$$\text{whence, } \frac{1}{17} = .0588235294\tfrac{2}{17}:$$

$$\text{again, } \frac{2}{17} = .0588235294\tfrac{2}{17} \times 2 = .1176470588\tfrac{4}{17},$$

$$\text{which gives } \frac{1}{17} = .\dot{0}58823529411764\dot{7}0588\tfrac{4}{17},$$

$$\text{and therefore } \frac{5}{17} = .\dot{2}94117647058823\dot{5}2941\tfrac{3}{17},$$

whereof the period comprises 16 figures: and this process proves that in *such instances* the period consists of the *same digits* whatever the numerator of the fraction may be, and merely commences with a different figure.

Again, we have $\frac{1150}{333} = \dot{3}.4\dot{5}$, the period of which is a *mixed* quantity; but for the application of the preceding rules, we may put it in the form $\frac{1150}{333} = 3.\dot{4}5\dot{3}$, whereof the period is entirely *fractional*.

Examples for Practice.

(1) What are the recurring decimals corresponding to the vulgar fractions,

$$\frac{2}{9},\ \frac{3}{11},\ \frac{13}{99},\ \frac{5}{37},\ \frac{129}{55}?$$

Answers: $.\dot{2}$, $.\dot{2}\dot{7}$, $.\dot{1}\dot{3}$, $.\dot{1}3\dot{5}$, $2.3\dot{4}\dot{5}$.

(2) Convert $\frac{4}{13}$, $\frac{5}{41}$ and $\frac{8}{53}$ into recurring decimals.

Answers: $.\dot{3}0769\dot{2}$, $.\dot{1}219\dot{5}$ and $.\dot{1}50943396226\dot{4}$.

(3) Find the vulgar fractions equivalent to the recurring decimals: $.\dot{5}$, $.\dot{0}2\dot{7}$, $.\dot{5}3\dot{4}$, $.\dot{4}26\dot{3}$.

Answers: $\frac{5}{9}$, $\frac{1}{37}$, $\frac{178}{333}$, $\frac{1421}{3333}$.

(4) What vulgar fractions will represent the values of $.36\dot{2}\dot{1}$, $.47\dot{5}4\dot{3}$, $.6\dot{7}6190\dot{4}$, $.008\dot{4}971\dot{3}$?

Answers: $\frac{239}{660}$, $\frac{3958}{8325}$, $\frac{71}{105}$, $\frac{83}{9768}$.

(5) Required the least numbers of which $.\dot{4}7619\dot{0}$ is the recurring quotient: and find the error in the corresponding fraction when .47619 is taken to represent it.

Answers: 10 and 21, and $\frac{1}{233331}$.

(6) Multiply $.\dot{5}7142\dot{8}$ by 1.25 and divide $.\dot{4}6153\dot{8}$ by 30.

Answers: $.\dot{7}1428\dot{5}$ and $.0\dot{1}5384\dot{6}$.

(7) Find the sum, difference, product and quotient of $.96\dot{3}4\dot{5}$ and $.\dot{3}$.

Answers:

the sum $= 1.29\dot{6}7\dot{8}$, the difference $= .63\dot{0}1\dot{2}$,

the product $= .32\dot{1}1\dot{5}$, the quotient $= 2.89\dot{0}3\dot{6}$.

(8) Find the value of $.97291\dot{6}$ of £1. and of $.013\dot{8}$ of 3.5 moidores of 27*s.* each.

Answers: 19*s.* 5½*d.* and 1*s.* 3¾*d.*

(9) What is the value of $.\dot{2}3\dot{4}$, when the unit is worth £20. and the worth of $.\dot{3}$ of $.\dot{3}$, when the unit is valued at £108?

Answers: £4. 13. $8\frac{1}{4}d.$ $\frac{11}{37}f.$ and £12.

(10) Find the value of $1.91\dot{6}$ of 8*s.* and of $3.0\dot{7}$ of 11*s.* 3*d.*

Answers: 15*s.* 4*d.* and £1. 14*s.* $7\frac{1}{2}d.$

(11) Determine the value of $46.\dot{9}\dot{0}$ of 1mi. 6fur. 20po. and of $13.2\dot{7}\dot{5}$ of $5\frac{1}{2}$ acres.

Answers: 85mi. 7po. $1\frac{1}{2}$yd. and 73ac. 2po. $20\frac{1}{6}$yds.

(12) Reduce $1.\dot{7}85714\dot{2}$ of a cwt. to its proper value, and find what decimal it is of a ton.

Answers: 1cwt. 3qrs. 4lbs. and $.089\dot{2}8571\dot{4}$.

(13) Required the exact value of $.\dot{7}$ of [illegible] 6*d.* $-.\dot{8}\dot{4}$ of 16*s.* 6*d.* $+.9\dot{2}\dot{7}$ of £2. 10*s.* 5*d.*

Answer: £1. 18*s.* 7*d.*

(14) Find the value of $.208\dot{3}$ of $.3\dot{4}2857\dot{1}$ of a cwt. and of $.\dot{8}4615\dot{3}$ of $.0\dot{8}\dot{1}$ of £6. 10*s.*

Answers: 8lbs. and 9*s.*

(15) Find the value of $.\dot{2}8571\dot{4}$ of 30£ + $6.\dot{8}5714\dot{2}$£ + $.\dot{6}$ of $.\dot{7}1428\dot{5}$ of $.\dot{6}$£ + $1.\dot{3}$ of $.\dot{4}2857\dot{1}s.$

Answer: £15. 15*s.* $10\frac{2}{7}d.$

CHAPTER VI.

RATIO AND PROPORTION,

WITH THEIR MOST IMPORTANT APPLICATIONS.

RATIO.

122. Def. 1. *Ratio* is the relation which one number has to another, or, which one *quantity* numerically considered bears to another of the *same kind*, the comparison being made by observing what *multiple, part* or *parts,* the former is of the latter.

Thus, the relation of the *abstract numbers* 4 and 2 is *written* 4 : 2, and *read four* to *two;* and it will be *expressed* by $\frac{4}{2}=2$: the same being used to denote the relation of the *concrete quantities* 4 and 2, provided they be of the *same kind* and of the *same denomination.*

123. Def. 2. Of the numbers or quantities *compared* and called the *Terms* of the ratio, the former is styled the *Antecedent* and the latter the *Consequent;* also, the ratio is said to be a ratio of *greater* or *less Inequality* according as the antecedent is *greater* or *less* than the consequent, and it is a ratio of *Equality* when these terms are *equal.*

Thus, 6 : 5 is a ratio of greater inequality; 4 : 9 is one of less inequality, and a ratio of equality may be denoted by 1 : 1, or 2 : 2, or 3 : 3, &c., at pleasure.

124. Hence, the *Magnitude* of a ratio is expressed by the vulgar fraction whereof the antecedent is the *Numerator* and the consequent the *Denominator;* thus, the ratio of £9 and £12, written 9 : 12, will have its magnitude expressed by the fraction $\frac{9}{12}$, or, reduced to its lowest terms, by the fraction $\frac{3}{4}$: whereas, the ratio of 9*d.* to 6*s.* will be that of 9*d.* to 72*d.*, which $=\frac{9d.}{72d.}=\frac{9}{72}=\frac{1}{8}$: and this is therefore the same as that of 9lbs. to 72lbs.

Also, if the terms of the ratio be vulgar fractions or decimals, the fraction expressing its magnitude may be simplified by the rules already given.

125. The magnitudes of two or more ratios may therefore be compared, by comparing the values of the vulgar fractions which represent them, according to the principle of the last Article.

If the ratios be $3:4$ and $5:7$; then their magnitudes will be represented by $\frac{3}{4}$ and $\frac{5}{7}$;

$$\text{but } \frac{3}{4}=\frac{21}{28} \text{ and } \frac{5}{7}=\frac{20}{28};$$

and $\frac{21}{28}$ being greater than $\frac{20}{28}$, it follows that the ratio $3:4$ is greater than the ratio $5:7$; in other words, 3 has to 4 a greater ratio than 5 has to 7.

126. *A Ratio of greater inequality is diminished, and a Ratio of less inequality is increased, by adding the same quantity to both its terms.*

First, let us take the ratio of *greater* inequality $7:5$, and add 1 to both its terms, so that it becomes $8:6$;

$$\text{then the original ratio} = \frac{7}{5} = \frac{42}{30},$$

$$\text{and the new ratio} = \frac{8}{6} = \frac{40}{30};$$

therefore the new ratio is *less* than the original one.

Secondly, taking the ratio of *less* inequality $8:11$, and adding 2 to each term, so as to make it $10:13$, we have

$$\text{the original ratio} = \frac{8}{11} = \frac{104}{143},$$

$$\text{and the new ratio} = \frac{10}{13} = \frac{110}{143},$$

the latter of which fractions being greater than the former, the new ratio is the *greater* of the two.

Exactly in the same manner it may be shewn that a Ratio of *greater inequality* is *increased*, and a Ratio of *less inequality* is *diminished*, by subtracting the same quantity from each of its terms.

127. *If the terms of a Ratio be multiplied or divided by the same quantity, the magnitude of the ratio will not be altered.*

Let the ratio be $3 : 8$; then its magnitude is $\frac{3}{8}$ which is equivalent to

$$\frac{6}{16}, \text{ or } \frac{9}{24}, \text{ or } \frac{12}{32}, \text{ \&c.}:$$

that is, the ratio $3 : 8$ is equal to each of the ratios $6 : 16$, $9 : 24$, $12 : 32$, &c., which arise from the equal *multiplication* of its terms: and conversely, each of the latter ratios is reducible to the original one by the equal *division* of its terms.

128. Def. 3. If the antecedents of two or more ratios be multiplied together for a new antecedent, and their consequents be multiplied together for a new consequent, the resulting ratio is said to be *compounded* of the others and it is called their *Compound Ratio.*

Thus, if the ratios be $2 : 3$, $4 : 7$ and $8 : 13$, the ratio which arises from their composition will be

$$2 \times 4 \times 8 : 3 \times 7 \times 13, \text{ or } 64 : 273.$$

Examples for Practice.

(1) What are the simplest expressions of the magnitudes of the ratios $3 : 5$, $4 : 12$ and $9 : 21$?

Answers: $\frac{3}{5}$, $\frac{1}{3}$ and $\frac{3}{7}$.

(2) Which of the ratios is greater, $5 : 9$ or $7 : 11$; $10 : 17$ or $17 : 23$ and $34 : 27$ or $37 : 31$?

Answers: $7 : 11$, $17 : 23$ and $34 : 27$.

(3) Which of the three ratios $7 : 15$, $1\frac{3}{4} : 2\frac{5}{8}$ and $.75 : .96$ is the greatest?

Answer: $.75 : .96$.

(4) Find whether the ratios $7 : 9$, $11 : 17$ and $10 : 7$ are increased or diminished by adding 1, 2, 3, to their terms respectively.

Answers: The first and second are increased and the third is diminished.

(5) Are the ratios 4 : 3, 9 : 13 and 15 : 22 increased or diminished by subtracting 2, 3, 4, from their terms respectively?

Answers: The first is increased and the second and third are diminished.

(6) What are the ratios arising from the composition of 5 : 12 and 6 : 25, and of 5 : 7, 7 : 18 and 18 : 35?

Answers: 1 : 10 and 1 : 7.

PROPORTION.

129. Def. 1. *Proportion* is the relation of *Equality* subsisting between *two* or *more* ratios.

Thus, the ratios 2 : 3 and 6 : 9, being expressible by the equal fractions $\frac{2}{3}$ and $\frac{6}{9}$, are *equal*, and the four numbers 2, 3, 6, 9 form a proportion which is *written*

$$2 : 3 :: 6 : 9,$$

and is *read*

as 2 is to 3 so is 6 to 9,

the numbers 2, 3, 6, 9 being its *Terms* which taken in order are called *Proportionals*.

Hence, in every proportion, the first term is greater than, equal to, or less than the second, according as the third term is greater than, equal to, or less than the fourth.

130. Def. 2. In a proportion thus expressed, the numbers 2 and 9 are called the *Extremes* and the numbers 3 and 6 the *Means:* and it follows immediately from the equality of the ratios denoted by

$$\frac{2}{3}=\frac{6}{9},$$

and the multiplication of them both by 27, that

$$\frac{2}{3}\times 27=\frac{6}{9}\times 27;$$

that is, $2\times 9=6\times 3$:

in words, if *four* numbers constitute a proportion, the product of the *extremes* is equal to the product of the *means.*

131. This property of a proportion proves immediately that *either* of the extremes may be obtained by

dividing the product of the means by the *other;* and that *either* of the means may be had by the division of the product of the extremes by the *other:* also, these qualities constitute the general practical application of Proportion.

132. The terms of a proportion may be made to undergo changes and modifications in the same way as the *corresponding* terms of the vulgar fractions.

Thus, $3 : 4 :: 9 : 12$, gives $3 \times 12 = 4 \times 9$: whence

$$\frac{3}{9} = \frac{4}{12}; \text{ or } 3 : 9 :: 4 : 12;$$

$$\frac{4}{3} = \frac{12}{9}; \text{ or } 4 : 3 :: 12 : 9;$$

and we observe that in each of these the product of the extremes equals that of the means.

Also, if four numbers form a proportion and any equi-multiples whatever of the first and second be taken, and any equi-multiples whatever of the third and fourth, the resulting numbers taken in order will still form a proportion.

For, since $5 : 3 :: 15 : 9$, or $\frac{5}{3} = \frac{15}{9}$; and also $\frac{2}{2} = \frac{3}{3}$;

$$\text{we have } \frac{5}{3} \times \frac{2}{2} = \frac{15}{9} \times \frac{3}{3}, \text{ or } \frac{5 \times 2}{3 \times 2} = \frac{15 \times 3}{9 \times 3};$$

$$\text{whence, } 5 \times 2 : 3 \times 2 :: 15 \times 3 : 9 \times 3.$$

Again, if any equi-multiples whatever of the first and third numbers be taken, and also any equi-multiples whatever of the second and fourth, the numbers thence arising will form a proportion.

Thus, if we take the proportion above, we have

$$\frac{5}{3} \times \frac{4}{7} = \frac{15}{9} \times \frac{4}{7}, \text{ or } \frac{5 \times 4}{3 \times 7} = \frac{15 \times 4}{9 \times 7};$$

$$\text{whence, } 5 \times 4 : 3 \times 7 :: 15 \times 4 : 9 \times 7.$$

The *new* ratios constituting these proportions being *equal* to the *original*, the division of the terms of a proportion, in accordance with this Article, will often

facilitate practical computations by diminishing the number of figures employed.

133. If four quantities of the *same* kind taken in order be proportionals, it will be *useful* to recollect that,

(1) The first : the third :: the second : the fourth.

(2) The second : the first :: the fourth : the third.

(3) The sum of the first and second : the first :: the sum of the third and fourth : the third.

(4) The sum of the first and second : the second :: the sum of the third and fourth : the fourth.

(5) The difference of the first and second : the first :: the difference of the third and fourth : the third.

(6) The difference of the first and second : the second :: the difference of the third and fourth : the fourth.

(7) The sum of the first and second : the difference of the first and second :: the sum of the third and fourth : the difference of the third and fourth.

These may easily be shewn to be correct by any of the proportions hitherto given.

134. Of two or more proportions if the corresponding terms be multiplied together, the numbers thence arising will also form a proportion.

Thus, if the proportions be

$$3 : 7 :: 6 : 14 \quad \text{and} \quad 4 : 9 :: 12 : 27;$$

$$\text{then} \quad \frac{3}{7} = \frac{6}{14} \quad \text{and} \quad \frac{4}{9} = \frac{12}{27};$$

$$\text{whence,} \quad \frac{3}{7} \times \frac{4}{9} = \frac{6}{14} \times \frac{12}{27}, \quad \text{or} \quad \frac{3 \times 4}{7 \times 9} = \frac{6 \times 12}{14 \times 27};$$

$$\text{and } 3 \times 4 : 7 \times 9 :: 6 \times 12 : 14 \times 27.$$

This operation is called the *Compounding* of proportions, and the resulting proportion is said to be *compounded* of the others.

In these Articles *abstract* nnmbers have been considered: but when the quantities are *concrete*, we must take care to exclude such proportions as *express* ratios between things of *different kinds:* thus, the ratio of

10lbs. to 15lbs. being the *same* as that of 2*s*. to 3*s*., we have the proportion

10lbs. : 15lbs. :: 2*s*. : 3*s*.;

but we cannot have the proportion

10lbs. : 2*s*. :: 15lbs. : 3*s*.

as no ratio subsists between 10lbs. and 2*s*. or between 15lbs. and 3*s*.

Nor indeed can we even in the first of these forms multiply together the *concrete* quantities so that the product of the extremes equals the product of the means; but what we do in finding any term in such cases, is to consider merely their *numerical* values, because the ratios being *abstract magnitudes* will remain the same whatever be the *nature* of the quantities they are used to compare. See the *Appendix*.

135. *Ratio* and *Proportion* as here used are generally called *Geometrical* Ratio and *Geometrical* Proportion, because they are employed in *Geometry* in the same sense: also, *Arithmetical* Ratio and *Arithmetical* Proportion are *sometimes* used to express the *Differences* of two or more numbers and their relations to each other, exactly in the same manner as we have throughout applied Ratio and Proportion to denote their *Quotients* and the relations subsisting among two or more such.

Thus, of 7 and 5, the *geometrical* ratio is $7 : 5 = \frac{7}{5} = 1\frac{2}{5}$: whereas their *arithmetical* ratio is $7 - 5 = 2$: also, the numbers 3, 4, 15, 20 form a *geometrical* proportion, because $\frac{3}{4} = \frac{15}{20}$: but 4, 3, 2, 1 constitute an *arithmetical* proportion, since $4 - 3 = 2 - 1$.

When necessary, the relations of numbers considered in the latter point of view may be determined by means of the equality $4 - 3 = 2 - 1$, in a manner similar to what has been done above.

136. If three numbers as 18, 13 and 8 be in what is called *continued* Arithmetical proportion, then $18 - 13 = 13 - 8$; and if $13 + 8$ be added to both members of this equality, we shall have

$$18 + 8 = 13 + 13;$$

that is, the *Sum* of the *Extremes* is equal to *twice* the *Arithmetical Mean* between them; and therefore the arithmetical mean is equal to *half* their sum.

In the same manner, 16, 8 and 4 are said to be in *continued* geometrical proportion, because

$$16 : 8 :: 8 : 4, \text{ or } \frac{16}{8} = \frac{8}{4};$$

and multiplying both sides of this equality by 8×4, we obtain

$$16 \times 4 = 8 \times 8;$$

or, the *Product* of the *Extremes* is equal to the *Square* of the *Geometrical Mean* between them: and consequently the geometrical mean between two numbers is equal to the *Square Root* of their product.

These *terms* and the corresponding *operations* form the chief substance of the next Chapter, and they have been noticed in this only because they appear to arise immediately out of what has been considered in it.

Applications of Ratio and Proportion.

137. Ratio and Proportion will now be applied and exemplified under the following heads.

(1) *The Rule of Proportion.*
(2) *Simple and Compound Interest.*
(3) *The Natures and Transfers of Stocks.*
(4) *Discount or Rebate.*
(5) *The Equation of Payments.*
(6) *The Rule of Fellowship.*
(7) *The Rule of Alligation.*
(8) *The Doctrine of Exchanges.*

I. THE RULE OF PROPORTION.

138. Def. As has been observed in *The Rule of Three* of which this is only another name, we have here *three* quantities either simple or compound given to find a *fourth* which shall complete the proportion; and this is called a *fourth proportional* to the three quantities proposed, taken in order.

139. Assuming as an *Axiom*, that *Effects* have the same *relation* or *ratio* to each other as the *Causes* which produce them under the same circumstances, it is evident that in any two cases of the same kind we shall have the following proportion:

First Cause : Second Cause :: First Effect : Second Effect;

and then, what was said in Articles (130) and (131) will enable us to find any *one* term if the *three* others be supposed to be given.

To avoid the trouble of writing the name of the *required* term or quantity at length, we shall always denote it by the simple *symbol* x which must be treated in the same way as any other number: and it may occupy any place in the proportion either by *itself* or as a *factor* either *integral* or *fractional* with given numbers, as in the following Examples.

Ex. 1. If 5 men can mow 12 acres of grass in a certain time; how many acres will 16 men be able to mow in the same or an equal time?

Here, $\left.\begin{array}{r}5 \text{ men} \\ 16 \text{ men}\end{array}\right\}$ are the first and second $\left\{\textit{Causes}:\right.$

$\left.\begin{array}{r}12 \text{ acres} \\ x \text{ acres}\end{array}\right\}$ are the first and second $\left\{\textit{Effects}:\right.$

whence we have the following proportion:

$$\overset{\text{men}}{5} : \overset{\text{men}}{16} :: \overset{\text{ac.}}{12} : \overset{\text{ac.}}{x};$$

and therefore by the Articles just referred to, we find

$$5 \times x = 16 \times 12 = 192:$$

$$\text{whence, } x = \frac{192}{5} \text{ ac.} = 38 \text{ ac. } 1 \text{ ro. } 24 \text{ po.}$$

Ex. 2. If 8oz. of bread be sold for 6*d.* when wheat is at £15. a load; what should be the price of wheat when 12oz. are sold for 4*d.*?

If the price of a load of wheat be *regulated* by, so as to be *proportional* to, the price of an ounce of bread, since,

$$\text{in the former case the price of 1oz.} = \tfrac{6}{8}d. = \tfrac{3}{4}d.,$$

$$\text{and in the latter the price of 1oz.} = \tfrac{4}{12}d. = \tfrac{1}{3}d.,$$

we must have following proportion,

$$\tfrac{3}{4}d. : \tfrac{1}{3}d. :: £15 : £x;$$

whence, $x = \frac{1}{3} \times 15 \div \frac{3}{4} = £\frac{20}{3} = £6.\ 13s.\ 4d.$, which is the required price.

These examples, the causes in which are *simple* terms being dependent upon only *one* magnitude, are instances of what is called *Direct Proportion*, because the effect is *greater* or *less* in the same proportion as the cause is *greater* or *less*.

Ex. 3. If 10 men can perform a piece of work in 12 days; how many days will it take 8 men to do the same?

Here, the causes will evidently be to each other as 10×12 to $8 \times x$; and the effects are the *same*, and may therefore be represented by 1, or any other symbol:

$$\text{whence, } 10 \times 12 : 8 \times x :: 1 : 1;$$

$$\text{therefore } 8 \times x = 10 \times 12 = 120,$$

$$\text{and } x = \frac{120}{8} = 15 \text{ days.}$$

Ex. 4. How much in length, that is 3ft. 9in. broad, will be equal to what is 37ft. 9in. long, and 7ft. 6in. broad?

Here, the first cause $= 45\text{in.} \times x\text{in.}$;

the second cause $= 90\text{in.} \times 453\text{in.}$;

and the effects are to be equal:

$$\text{therefore } 45 \times x : 90 \times 453 :: 1 : 1;$$

$$\text{whence, } 45 \times x = 90 \times 453,$$

$$\text{and } x = \frac{90 \times 453}{45} = 906\text{in.} = 75\text{ft. } 6\text{in.}$$

In these two examples, the *entire* causes are *compound* quantities, depending upon two *subordinate* causes; and because the effect is the *same*, each subordinate cause is *less* or *greater* according as the other is *greater* or *less*, constituting what is called *Inverse Proportion*.

Ex. 5. If a person can perform a journey of 100 miles in 12 days of 8 hours each; how far will he be able to travel in 15 days of 9 hours each?

Here, 12×8 and 15×9 are the causes, and the distances travelled 100 and x are the effects: whence,

$$12 \times 8 : 15 \times 9 :: 100 : x;$$

$$\text{and } x = \frac{15 \times 9 \times 100}{12 \times 8} = 140\tfrac{5}{8} \text{ miles.}$$

Ex. 6. If 60 bushels of corn feed 6 horses for 50 days; in how many days will 15 horses consume 75 bushels?

The causes are 6×50 and $15 \times x$, and the effects are 60 and 75 bushels: therefore

$$6 \times 50 : 15 \times x :: 60 : 75,$$

$$\text{or, } 2 \times 10 : x :: 4 : 5;$$

$$\text{whence, } x = \frac{2 \times 10 \times 5}{4} = 25 \text{ days.}$$

In the former of these examples, the distances travelled are in the *compound* ratio of the numbers of days and their lengths: and in the latter, the numbers of bushels have the same ratio as that which is *compounded* of the numbers of horses and days.

Ex. 7. If 25 labourers can dig a trench 220 yards long, 3ft. 4in. wide and 2ft. 6in. deep, in 32 days of 9 hours each: how many would it require to dig a trench half a mile long, 2ft. 4in. deep and 3ft. 6in. wide, in 36 days of 8 hours each?

$$\left.\begin{array}{l} \text{First cause} = 25 \times 32 \times 9 \\ \text{second cause} = x \times 36 \times 8 \end{array}\right\} \text{being the products of the } \textit{subordinate} \text{ causes:}$$

$$\left.\begin{array}{l} \text{first effect} = 220 \times \frac{10}{9} \times \frac{5}{6} \\ \text{second effect} = 880 \times \frac{7}{9} \times \frac{7}{6} \end{array}\right. \text{the } \textit{mixed} \text{ quantities being reduced to fractions of 1 yard.}$$

Hence, we have the following proportion:

$$25 \times 32 \times 9 : x \times 36 \times 8 :: 220 \times \frac{10}{9} \times \frac{5}{6} : 880 \times \frac{7}{9} \times \frac{7}{6};$$

$$\text{or, } 25 : x :: 1 \times 10 \times 5 : 4 \times 7 \times 7;$$

$$\text{whence, } x = \frac{25 \times 4 \times 7 \times 7}{1 \times 10 \times 5} = 98 \text{ labourers.}$$

These examples, the causes and effects being simple and compound quantities consisting of their respective

subordinate *partial* causes and effects, are instances of *Compound Proportion* in its fullest meaning.

Ex. 8. If 10 excavators can dig 12 loads of earth in 16 hours, whilst 12 others can dig 9 loads in 15 hours: find the time in which they will jointly dig 100 loads.

Since, the ratio $10 \times 16 : 12 \times 15$ is *not* equal to the ratio $12 : 9$, it follows that the individuals of the two sets do not work at the *same rate;* but the rate of one of the *first* set being represented by $\frac{12}{10 \times 16} = \frac{3}{40}$, that of one of the *second* set will be equal to $\frac{9}{12 \times 15} = \frac{1}{20}$:

whence, $\left\{10 \times \frac{3}{40} + 12 \times \frac{1}{20}\right\} \times$ the required time $= 100$;

or, $\left(\frac{3}{4} + \frac{3}{5}\right)$ of the required time $= 100$ hours:

that is, the required time $= \frac{20}{27}$ of 100hrs $= 74\frac{2}{27}$hrs.

140. In practice, when the *partial* causes and effects consist of compound quantities, it is most convenient to express them by vulgar fractions or decimals: and when the *entire* causes and effects are compound quantities, to proceed as in the *third* chapter (all the examples of which are instances of this rule) shortening the operation as much as possible by means of Article (132).

Examples for Practice.

(1) Find a number which shall have the same ratio to 7 as 27 has to 3: also, a magnitude to which 39 has the same relation as $3\frac{1}{4}$ has to $2\frac{3}{5}$.

Answers: 63 and $31\frac{1}{5}$.

(2) Required the number which has to 40, the ratio of 3.75 to 3: and find a fourth proportional to $\frac{2}{7}$, 17 and 1.25.

Answers: 50 and $74\frac{3}{8}$.

(3) Complete the proportion of which the first, second and fourth terms are $\frac{1}{20}$, 35 and $3\frac{3}{4}$: also, that

whose first, third and fourth terms are .35, 125 and .0145.

Answers: The third term is $\frac{3}{560}$ and the second term is .0000406.

(4) If when malt costs 63*s.* a quarter the price of a quart of ale be $4\frac{1}{2}d.$, what should its price be when malt is at 66*s.* 6*d.* per quarter?

Answer: $4\frac{3}{4}d.$

(5) If a person can perform a journey in 24 days of $10\frac{1}{2}$ hours each; what time will it take him to do the same when the days are $12\frac{3}{4}$ hours long?

Answer: $19\frac{13}{17}$ days.

(6) How much in length that is 15 poles in breadth, will be equivalent to an acre of land which is 40 poles in length and 4 poles in breadth?

Answer: 10po. 3yds. 2ft.

(7) If 400 soldiers consume 5 barrels of flour in 12 days; how many soldiers will consume 15 barrels in 2 days?

Answer: 7200 soldiers.

(8) If 7lbs. of sugar be sold for 4*s.* 8*d.* when the cost of a cwt. is £3. 7*s.* 8*d.*: what should be the cost of a cwt. when 11lbs. is sold for 7*s.* $1\frac{1}{4}d.$?

Answer: £3. 5*s.* $6\frac{1}{2}d.$ $\frac{1}{2}f.$

(9) If a sixpenny loaf weigh 4.35lbs. when wheat is at 5.75*s.* a bushel, what must be paid for 49.3lbs. of bread when wheat is at 18.4*s.* a bushel?

Answer: 18.3*s.*

(10) A person is able to travel 142.2 miles in $4\frac{1}{2}$ days of 10.164 hours each: in how many days of 8.4 hours each can he travel 505.6 miles?

Answer: 19.36 days.

(11) If 20 men can perform a piece of work in 12 days; how many men will perform another piece of work three times as great in a fifth part of the time?

Answer: 300 men.

(12) If 12 men can mow a field 300 yards square in 10 days: how many men can mow a field of 600 yards in length and 10 yards in breadth, in 4 days?

Answer: 2 men.

(13) If 3 men working 10hrs. a day can reap a field measuring 150yds. by 240yds. in 5 days, how many men working 12hrs. a day, can reap a field measuring 192yds. by 300yds. in 2 days?

Answer: 10 men.

(14) If 7 men can build a wall 245yds. long, 8ft. high and 18in. thick, in 35 days of 12hrs. each; what length of wall, 10ft. high and 27in. thick, could 12 men build in 43 days of 10hrs. each?

Answer: 229yds. 1ft.

(15) If 27 men can do a piece of work in 14 days, working 10 hours a day; how many hours a day must 24 boys work, in order to complete the same in 45 days, the work of a boy being half that of a man?

Answer: 7 hours.

(16) If 4 artillery men can fire a gun 48 times and 5 men 52 times in an hour; how much more time will be required for firing 21216 shots from 26 guns, when there are 4 men to a gun than when there are 5 men?

Answer: $1\frac{4}{13}$ hours.

(17) If 10 cannon which fire 3 rounds in 5 minutes, kill 270 men in $1\frac{1}{2}$ hours; how many cannon which fire 5 rounds in 6 minutes, will kill 500 men in 1 hour, at the same rate?

Answer: 20 cannon.

(18) If 120 men in 3 days of 12 hours each, can dig a trench 30 yds. long, 2ft. broad and 4ft. deep; how many men would be required to dig a trench 50yds. long, 6ft. deep and $1\frac{1}{2}$yds. broad, in 9 days of 15 hours each?

Answer: 180 men.

(19) If 6 men can reap 15 acres in 3 days of 14hrs. each and 10 boys can reap $10\frac{5}{7}$ acres in 5 days of 9hrs. each, find the ratio of the work of a man to that of a boy:

and determine what number of acres 4 men and 7 boys together reap in a day.

Answers: 5 : 2 and $4\frac{5}{8}$ acres.

(20) A watch, which is 10 minutes too fast at twelve o'clock on Monday, gains 3min. 10sec. per day; what will be the time by the watch at a quarter past ten in the morning of the following Saturday?

Answer: 40min. $36\frac{7}{48}$sec. past 10.

(21) Of two clocks, one gains 10 minutes and the other loses $7\frac{1}{2}$ minutes in 24hrs; what will be the difference of the times indicated by them at 6 o'clock on Friday morning, if they are together on the noon of the preceding Tuesday?

Answer: 48min. $7\frac{1}{2}$sec.

(22) If beer which is brewed with 3 bushels of malt to the barrel cost 1*s.* 3*d.* per gallon, when malt is at 62*s.* 8*d.* the quarter: how much will beer cost per gallon, which is brewed with 5 bushels of malt to the barrel, when a quarter of malt costs 50*s.*?

Answer: 1*s.* $7\frac{3}{4}$*d.* $\frac{87}{47}$*f.*

II. SIMPLE AND COMPOUND INTEREST.

141. Def. *Interest* is the payment made for the *use* of money lent for any length of time, being generally estimated at so much for £100. during a year and expressed by so much *per cent. per annum:* the money lent is called the *Principal*, the interest of £100. for a year the *Rate per cent.* and the sum lent together with its interest is termed the *Amount.*

It is called *simple* interest, when the money advanced only pays interest for the whole time it is lent; and *compound* interest, when, at the end of any *assigned* period, as a year for instance, the interest which has accrued is added to the principal, and the whole then bears interest at the same rate for another *equal* period, and so on.

Hence, it is evident that simple interest is proportional to the product of the sum lent, the rate of interest and the time.

142. *Simple Interest.*

Ex. Find the simple interest and amount of £237. 10. for $2\frac{1}{2}$ years, at 5 per cent. per annum.

From what has just been said, we have

$$\overset{\pounds.}{100} : \overset{\pounds.}{237} \cdot \overset{s.}{10} :: \overset{\pounds.}{5} : \overset{\pounds.}{x};$$

and therefore

$$x = \frac{\overset{\pounds.}{237} \cdot \overset{s.}{10} \times 5}{100} = \frac{\overset{\pounds.}{237.5} \times 5}{100} = \frac{\overset{\pounds.}{1187.5}}{100} = \overset{\pounds.}{11.875} = \overset{\pounds.}{11} \cdot \overset{s.}{17} \cdot \overset{d.}{6}$$

is the interest of $\overset{\pounds.}{237} \cdot \overset{s.}{10}$. for one year:

hence, $\overset{\pounds.}{11} \cdot \overset{s.}{17} \cdot \overset{d.}{6}. \times 2\frac{1}{2} = \overset{\pounds.}{29} \cdot \overset{s.}{13} \cdot \overset{d.}{9}$ is the interest of $\overset{\pounds.}{237} \cdot \overset{s.}{10}$ for $2\frac{1}{2}$ years: and therefore

$\overset{\pounds.}{237} \cdot \overset{s.}{10} + \overset{\pounds.}{29} \cdot \overset{s.}{13} \cdot \overset{d.}{9} = \overset{\pounds.}{267} \cdot \overset{s.}{3} \cdot \overset{d.}{9}$ is the amount in $2\frac{1}{2}$ years.

In practice, we work in the following *form:*

$$\begin{array}{r}
\overset{\pounds.}{237} \cdot \overset{s.}{10} \\
5 \\
\hline
\pounds 11 \cdot 87 \cdot 10 \\
20 \\
\hline
s17 \cdot 50 \\
12 \\
\hline
d6 \cdot 00
\end{array}$$

where the sum proposed is multiplied by the rate per cent., and from the right of each successive denomination, *two* figures are cut off instead of dividing by 100: whence, we have

$$\begin{array}{c|c|rrrl}
\frac{1}{2} & \frac{1}{2} & \overset{\pounds.}{11} & \cdot\ \overset{s.}{17} & \cdot\ \overset{d.}{6} & = \text{interest for 1 year:} \\
 & & & & 2\frac{1}{2} & \\
\hline
 & & 23 & \cdot\ 15 & \cdot\ 0 & = \text{interest for 2 years:} \\
 & & 5 & \cdot\ 18 & \cdot\ 9 & = \text{interest for } \frac{1}{2} \text{ year:} \\
\hline
 & & 29 & \cdot\ 13 & \cdot\ 9 & = \text{interest for } 2\frac{1}{2} \text{ years:} \\
 & & 237 & \cdot\ 10 & \cdot\ 0 & = \text{principal:} \\
\hline
 & & \pounds 267 & \cdot\ 3 & \cdot\ 9 & = \text{amount in } 2\frac{1}{2} \text{ years:}
\end{array}$$

and this *form* gives rise to the following Rule.

Rule for Simple Interest.

Multiply the principal by the rate per cent. by Compound Multiplication or Practice: from the pounds in the product cut off *two* figures to the *right*, and the remaining figures will be the pounds of the interest: reduce the figures cut off to shillings, taking in the shillings of the said product, cut off as before, and thus proceed; and the interest for one year will be obtained: multiply this interest by the number of years proposed, whether whole or fractional, and the required interest will be found.

If the interest for 1 year be not required, we may multiply the sum proposed by the product of the rate per cent. and the number of years, and *then* cut off, &c.

When the interest for *months* and *days* is required, it is found by *Practice* and the *Rule of Three* respectively, reckoning 12 months and 365 days to a year: but if *calendar* months be specified, the interest is accurately determined by finding the number of days they contain and proceeding by the Rule of Three.

143. *Commission, Brokerage, Insurance, &c.*, being charges of certain sums *per cent.*, amount to the same thing as the interest for one year at the same rate, and they may therefore be computed by the same rule.

Examples for Practice.

(1) Find the simple interest of £382. 10*s.* for 1 year, at 5 per cent.

Answer: £19. 2*s.* 6*d.*

(2) Required the amount of £537. 16*s.* 8*d.* in 4 years, at $2\frac{1}{2}$ per cent. simple interest.

Answer: £591. 12*s.* 4*d.*

(3) Find the amount of £325. 16*s.* 8*d.* at $4\frac{1}{4}$ per cent. simple interest, in $3\frac{1}{2}$ years.

Answer: £374. 6*s.* $0\frac{1}{4}$*d.*

(4) What is the amount of £345. 17*s.* 6*d.* in 3 years, at 4 per cent. simple interest?

Answer: £387. 7*s.* $7\frac{1}{5}$*d.*

(5) Determine the amount of £635. 18*s*. 4½*d*. in 3½ years, at 3 per cent. simple interest.

Answer: £702. 13*s*. 9½*d*.$\frac{61}{100}$*f*.

(6) Required the amount of £825. 13*s*. 8*d*. at 4¾ per cent. simple interest, in 3 years and 5 months.

Answer: £959. 13*s*. 8¼*d*.$\frac{139}{200}$*f*.

(7) What is the interest of £535. for 117 days, at 4¾ per cent.?

Answer: £8. 2*s*. 11$\frac{8}{365}$*d*.

(8) Find the simple interest of £960. 12*s*. 6*d*. for 5 years, 8 months and 73 days, at 3¼ per cent.

Answer: £183. 3*s*. 2$\frac{1}{5}$*d*.

(9) What is the interest of £240. from January 7 to September 12, 1849, at 4 per cent.?

Answer: £6. 10*s*. 5¼*d*. $\frac{303}{365}$*f*.

(10) Find the amount of £237. 10*s*. in 2 years, 8 months and 29 days, at 5 per cent. simple interest.

Answer: £270. 2*s*. 2¼*d*. $\frac{55}{73}$*f*.

(11) Ascertain the commission or brokerage on £832., at 2½ per cent.

Answer: £20. 16*s*.

(12) A broker procures an insurance upon property to the amount of £1850; what will he have for his trouble at 3*s*. 9*d*. per cent.?

Answer: £3. 9*s*. 4½*d*.

144. Of the four quantities, the *Principal*, the *Rate*, the *Time* and the *Interest*, when any *three* are given, the remaining *one* may be found by reversing the process: and exercises will be had immediately from the preceding examples, but they are not often required.

145. *Compound Interest.*

Ex. Required the compound interest of £250. for two years, at 5 per cent. per annum.

$$\begin{array}{r} \pounds. \\ 250 \\ 5 \\ \hline \pounds 12.50 \\ 20 \\ \hline s\,10.00 \end{array}$$

$$\begin{array}{rl} \pounds. \quad s. & \\ 250.\ \ 0 & = \text{first principal:} \\ 12.10 & = \text{interest for the first year:} \\ \hline \pounds 262.10 & = \text{amount in one year:} \end{array}$$

$$\begin{array}{r} \pounds. \quad s. \\ 262.10 \\ 5 \\ \hline \pounds 13.12.10 \\ 20 \\ \hline s\,2.50 \\ 12 \\ \hline d\,6.00 \end{array}$$

$$\begin{array}{rl} \pounds. \quad s. \quad d. & \\ 262.10.0 & = \text{second principal:} \\ 13.\ \ 2.6 & = \text{interest for the second year:} \\ \hline 275.12.6 & = \text{amount in two years:} \\ 250.\ \ 0.0 & = \text{original principal:} \\ \hline \pounds 25.12.6 & = \text{compound interest in two years:} \end{array}$$

and we see that *this* interest is the *sum* of the interests of the *first* and *second* years together, upon their respective principals.

Here the interest may be supposed to form part of the new principal at the ends of any *equal* intervals of time, as *half-yearly*, *quarterly*, &c.; but then the operation above must be repeated for *every* such interval: and when the compound interest is required for any number of years and parts of a year, it is inconsistent with the principles of the subject to suppose that the

interest becomes due at any other interval of time than what is expressed by the *primitive* fraction of which the parts are made up: as, for instance, when compound interest for *three-fourths* of a year is required, it is necessarily implied that the interest is due at the end of each *quarter*.

When the *equitable* principle just mentioned is not attended to, it is *customary* to find the interest for one year in addition to the number of *entire* years expressed, and then to take the part or parts of that interest which correspond with the proposed part or parts of a year, and to add it to the amount already obtained: but although this be not a bad approximation to the true amount, questions for the *exercise* of students should never be proposed which require its application.

The operation last given will preclude the necessity of laying down a rule in *words* for finding the amount and compound interest.

Examples for Practice.

(1) Required the amount of £350. in 3 years, at 5 per cent. compound interest.

Answer: £405. 3*s*. $4\frac{1}{2}$*d*.

(2) Find the compound interest of £540. in 2 years, at 4 per cent.

Answer: £44. 1*s*. $3\frac{1}{4}$*d*. $\frac{11}{25}$*f*.

(3) What is the compound interest of £150. in 4 years, at $2\frac{1}{2}$ per cent.?

Answer: £15. 11*s*. $5\frac{1}{4}$*d*. $\frac{9}{100}$*f*.

(4) To what sum will £725. accumulate in 4 years, at 5 per cent. compound interest?

Answer: £881. 4*s*. $10\frac{7}{80}$*d*.

(5) Determine the amount of £550. 10*s*. in a year and a half at 4 per cent. per annum, compound interest, the interest being due half-yearly.

Answer: £584. 3*s*. $10\frac{3}{4}$*d*. .20384*f*.

(6) Required the amount of £819. 4*s*. in 6 years, allowing £12. 10*s*. per cent. compound interest.

Answer: £1660. 15*s*. $0\frac{3}{4}$*d*.

The solutions may frequently be simplified by decimals, or by taking the *aliquot* parts of the sum proposed; but the processes cannot, as in the last Article, be reversed by the principles hitherto explained.

III. THE NATURES AND TRANSFERS OF STOCKS.

146. Def. The exigencies of a Country sometimes compel its governing body to *borrow*, or to *contract a Loan*, for the benefit of the public service: and this is effected by giving to the *Lenders* in exchange for their money, *Government Bonds* or *Acknowledgments*, implying that the Nation is indebted to them for the sums advanced, whilst it reserves to itself the option of the *Time* of paying off the *Principal*, on the express condition that the *Interest* is regularly discharged at the time fixed upon.

147. Any part of these bonds is *transferable* from one person to another at pleasure, and each bond is usually styled £100. *Stock*, bearing interest at a certain rate, the subdivisions of £1. stock being the same as those of *sterling* money.

Thus, in what are called the 3, $3\frac{1}{2}$ and 4 *per cent.* Stocks, one of these bonds entitles its owner to the sums of £1. 10*s.*, £1. 15*s.* and £2. respectively at the end of every *half-year* as interest: and that portion of the revenues of the country out of which the interest of these *Stocks* and the expenses connected with them are paid, is termed the *Funds.*

148. If a person *sell out* his stock from the Funds, he will be able to obtain more or less sterling money for each of his bonds, according to the interest it bears and also according to the circumstances of the times, which may influence the *stability* of the national credit: and if he *buy into* or *invest capital* in the Funds, the sum of ready money advanced by him for each bond will be regulated by the same circumstances.

149. When a *transfer* of capital is made from one kind of stock to another, it is evident that there will be an equitable claim for *more* or *fewer* bonds of the second stock, according as the rate of interest of such bonds is *less* or *greater* than that of the first: thus, a number of

bonds or *quantity of stock* in the 4 per cents., will produce the same interest as a *greater* quantity of stock in the 3 per cents., and consequently be of the same value to the possessor in point of income.

150. The same view is taken of money advanced to *Foreign Governments* or to *Trading Companies:* and the computations necessary in all *equitable* transactions of this kind must depend upon the *Rule of Proportion:* and those of most frequent occurrence will be explained in the subsequent examples.

Ex. 1. How much money must be paid for £2400. in the three per cent. *consols* (consolidated annuities), at $89\frac{1}{2}$ per cent.?

Here, we have the following proportion:

$$\begin{array}{ccccccc} \text{£.stock.} & & \text{£.stock.} & & \text{£.} & & \text{£.} \\ 100 & : & 2400 & :: & 89\frac{1}{2} & : & x; \end{array}$$

and the operation may be conducted in the *form* below:

$$\begin{array}{r l} \text{£}2400 & \\ 10 & \times 9 - \frac{1}{2} = 89\frac{1}{2} \\ \hline 24000 & \\ 9 & \\ \hline 216000 & \\ 1200 & \\ \hline 2148.00 & \end{array}$$

that is, £2148. *sterling* will purchase £2400. of *this* stock when it is at $89\frac{1}{2}$ per cent.

If we reverse the operation, we may find the quantity of stock at $89\frac{1}{2}$ which will be purchased for £2148. sterling, as follows:

$$\begin{array}{ccccccc} \text{£.} & & \text{£.} & & \text{£.stock.} & & \text{£.stock.} \\ 89\frac{1}{2} & : & 2148 & :: & 100 & : & x; \end{array}$$

$$\text{and therefore } x = \frac{2148 \times 100}{89\frac{1}{2}} = \text{£}2400.$$

Ex. 2. A person invests £3000. in the three per cents. when they are at $90\frac{1}{8}$; what amount of interest will he receive half-yearly?

Here, £$90\frac{1}{5}$ sterling produces £3. yearly or £$1\frac{1}{2}$. half-yearly: and therefore we have

$$\overset{\pounds.}{90\tfrac{1}{5}} : \overset{\pounds.}{3000} :: \overset{\pounds.}{1\tfrac{1}{2}} : \overset{\pounds.}{x};$$

$$\text{whence, } x = \frac{3000 \times 1\frac{1}{2}}{90\frac{1}{5}} = \frac{3000 \times 15}{902}$$

$$= \frac{1500 \times 15}{451} = \frac{22500}{451} = \pounds 49.\ 17s.\ 9\tfrac{1}{4}d.\ \tfrac{257}{451}f. = \text{his } \textit{dividend.}$$

Conversely, if a person wish to receive the last-mentioned sum half-yearly, he may invest £3000. in the three per cents. when they are at $90\frac{1}{5}$ for this purpose; since

$$\overset{\pounds.}{1\tfrac{1}{2}} : \overset{\pounds.}{49} . \overset{s.}{17} . \overset{d.}{9\tfrac{1}{4}} . \overset{f.}{\tfrac{257}{451}} :: \overset{\pounds.}{90\tfrac{1}{5}} : \overset{\pounds.}{3000};$$

but the same income might be acquired by the investment of a different sum of money in a different kind of stock, dependent upon the circumstances affecting it.

Ex. 3. At what rate will a person receive interest, who invests his capital in the 4 per cents. when they are at 104?

Since £104. sterling produces an interest of £4. annually, we have

$$\overset{\pounds.}{104} : \overset{\pounds.}{100} :: \overset{\pounds.}{4} : \overset{\pounds.}{x};$$

$$\text{and } x = \frac{100 \times 4}{104} = \frac{50}{13} = \pounds 3.\ 16s.\ 11\tfrac{1}{13}d. \text{ is the rate per cent.}$$

Conversely, when the interest of money is £3. 16*s*. $11\frac{1}{13}$*d*. per cent., the equitable value of the 4 per cent. stock is £104. sterling.

Ex. 4. A person transfers £1000. stock from the 4 per cents. at 90, to the 3 per cents. at 72: find how much of the latter stock he will hold, and the alteration made in his annual income.

Here, x denoting the quantity of the latter stock, we have

$$1000 \times 90 : x \times 72 :: 1 : 1;$$

$$\text{whence, } x = \frac{1000 \times 90}{72} = \frac{10000}{8} = \pounds 1250,$$

is the quantity of stock in the 3 per cents.:

also £1000. at 4 per cent. gives an income of £40.: and £1250. at 3 per cent. gives an income of £37. 10*s.*: therefore the *diminution* of his income is £2. 10*s.*

From this example, it appears to be *more advantageous* to invest in the 4 per cents. at 90 than in the 3 per cents. at 72, as will be seen also from the circumstance of $\frac{4}{90}$ being *greater* than $\frac{3}{72}$, which shews the reason at once.

151. *Purchases* and *Sales* of stock are made through *Agents* called *Stock-Brokers* at the rate of £$\frac{1}{8}$ or 2*s.* 6*d.* per cent. upon the stock *transferred*: these agents are employed by both *buyers* and *sellers*, and the *brokerage* must therefore be *added* to the price of stock which is *bought* and *subtracted* from the price of that which is *sold* through them: and this is done by *increasing* or *diminishing* the current price of £100. stock by £$\frac{1}{8}$.

152. *Stock-jobbing* is dealing in the Stocks with the view of gaining money by the rise and fall of the market price: and persons possessed of *property* may of course *buy* or *sell* stock, according as the price is likely to *rise* or *fall*, in the expectation of realizing a profit by the difference of price: but the *practice* of stock-jobbing may be explained as follows.

A agrees to sell to *B*, £1000. in the 3 per cents. for £870. to be transferred at the end of a *certain time*: *A* has indeed no such stock, but if its price on the *day of transfer* should be only 86 he may purchase what will enable him to fulfil his engagement for £860., and he will thus gain £10. by the transaction: on the contrary, if the price were 89, it would require £890. for the same purpose and he would then be a loser of £20. These are called *time bargains* or *bargains for the Account*, and are settled without the purchase of stock at all by payment of the *difference*: the *buyers* being known as *Bulls* pulling *down*, and the *sellers* as *Bears* forcing *up*, in the language of the *Stock Exchange*: but not being recognized by law, the principles by which this business is supported are merely a sense of *honour* or *disgrace* and the retention or loss of future *credit*.

153. It is the province of the *Chancellor of the Exchequer* to propose the terms of a *Government Loan*: and

if a person, asking 5 per cent. for his money, were to advance £1000. *money*, the nation would become indebted to him in £1666. 13*s*. 4*d*. *stock* in the 3 per cents. at *par*, because this sum produces the same interest. When the price of stock is *above* or *below* par, the calculations will be conducted as in the preceding pages; and in order to raise the *immense* sums that are frequently wanted, the *Contractors* or *Subscribers* are sometimes considered entitled to an advantage of one kind or other which is called a *Douceur* or *Bonus*.

Such stocks constitute what is termed *The funded Debt;* and whenever sums have been advanced to the government and have not been thus disposed of, they are styled *The unfunded Debt*.

Temporary loans are raised from time to time on what are called *Exchequer Bills* for single *hundreds* or *thousands* of Pounds, bearing a certain interest per cent. *per day*, from the days of their dates to the days when they are paid off.

Examples for Practice.

(1) What is the purchase of £5050. stock, at $85\frac{3}{8}$ per cent.?

Answer: £4311. 8*s*. 9*d*.

(2) If the 4 per cents. be at $82\frac{1}{8}$, what quantity of stock can be purchased for £821. 5*s*.?

Answer: £1000.

(3) A person invests £2000. in the 3 per cent. consols, when they are at $88\frac{1}{2}$: what annual income is he entitled to?

Answer: £67. 15*s*. $11\frac{11}{59}$*d*.

(4) What sum must be invested in the 3 per cents. at $77\frac{1}{2}$ to produce an income of £120. a year?

Answer: £3100.

(5) A person investing in the 4 per cents. receives $4\frac{3}{8}$ per cent. interest for his money: what is the price of the stock?

Answer: $91\frac{3}{7}$.

(6) How much stock can be purchased by the transfer of £1000. stock from the 3 per cents. at 72, to the

4 per cents. at 90: and what annual income will it produce?

Answers: £800 and £32.

(7) If I buy £650. stock in the 3 per cents. at $90\frac{3}{8}$, and pay $\frac{1}{8}$ for brokerage: what does it cost me?

Answer: £588. 5*s.*

(8) What sterling money shall I receive for £1760. 16*s.* 8*d.* stock at $90\frac{3}{8}$, and $\frac{1}{8}$ per cent. commission?

Answer: £1589. 3*s.* $0\frac{1}{2}$*d.*

(9) If a person invest £650. in the stocks at $76\frac{1}{2}$, and sell it immediately at $77\frac{31}{40}$; find his gain by the transaction.

Answer: £10. 16*s.* 8*d.*

(10) A person having £10000. in the 3 per cents. sells out at 65 and invests the produce in the 4 per cents. at $82\frac{1}{2}$: find the change in his income.

Answer: £15. 3*s.* $0\frac{5}{11}$*d.* increase.

(11) When a certain stock is at $97\frac{1}{2}$ a person possesses what would realize £879: find what quantity of another stock at $88\frac{1}{4}$ he ought to receive in exchange for it.

Answer: £$996\frac{12}{353}$.

(12) The sum of the dividends on a quantity of 3 per cent. stock for 13 years was £3081: how much stock was there, and what will be its sterling worth when the fund is sold at $79\frac{7}{8}$ per cent.?

Answers: £7900 and £$6310\frac{1}{8}$.

IV. DISCOUNT OR REBATE.

154. DEF. *Discount* or *Rebate* is an allowance or abatement made upon a debt discharged before it is due, in consideration of ready money, the simple interest of money being reckoned at a given rate: and when the discount is subtracted from any proposed sum, the remainder is termed the *Present Worth* or *Present Value.*

Ex. Find the present worth and discount of £121. 15*s.* due 5 months hence, at $3\frac{1}{2}$ per cent. simple interest.

Since £100 at $3\frac{1}{2}$ per cent. amounts to £101. 9*s.* 2*d.* in 5 months, it is evident that £101. 9*s.* 2*d.* due 5 months hence is of the same value as £100. ready money; and other sums for *this* time in the same proportion:

$$\begin{array}{lcccc} & \text{£. } s. \; d. & & \text{£. } s. & & \text{£.} \\ \text{whence,} & 101 \,.\, 9 \,.\, 2 & : & 121 \,.\, 15 & :: & 100 : x, \end{array}$$

$$\text{or, } 24350d. : 29220d. :: 100 : x;$$

therefore $x = \dfrac{2922}{2435} \times £100. = \dfrac{6}{5} \times £100. = £120.$ is the present worth: and the discount = the proposed sum − the present worth = £121. 15*s.* − £120. = £1. 15*s.*

Also, since £101. 9*s.* 2*d.* pays £1. 9*s.* 2*d.* discount for 5 months, we may find the discount at once; thus,

$$\begin{array}{ccccc} \text{£. } s. \; d. & & \text{£. } s. & & \text{£. } s. \; d. \\ 101 \,.\, 9 \,.\, 2 & : & 121 \,.\, 15 & :: & 1 \,.\, 9 \,.\, 2 : x; \end{array}$$

whence, $x = \dfrac{6}{5}$ of £1. 9*s.* 2*d.* = £1. 15*s.* is the discount: and the present worth = the proposed sum − the discount = £121. 15*s.* − £1. 15*s.* = £120. as before.

From these steps we derive the following Rules.

For the Present Worth. As the amount of £100. for the given time at the given rate : the proposed sum :: £100. : the present worth.

For the Discount. As the amount of £100. for the given time at the given rate : the proposed sum :: the interest of £100. for that time : the discount.

If no time be mentioned, the discount for a year is understood: and the following *forms* for the operations will be *practically* convenient:

mo.		£	s.	d.	
4	$\frac{1}{3}$	3 .	10 .	0	= interest of £100. for 12 months:
1	$\frac{1}{4}$	1 .	3 .	4	
		0 .	5 .	10	
		1 .	9 .	2	= interest of £100. for 5 months:
		100			
		101 .	9 .	2	= amount of £100. in 5 months:

$$\begin{array}{l} \quad\;\; £.\;\; s.\;\; d. \quad\;\; £.\;\; s. \qquad\; £. \\ \text{then, } 101 \,.\, 9 \,.\, 2 : 121 \,.\, 15 :: 100 : \text{the present worth;} \\ \qquad 101 \,.\, 9 \,.\, 2 : 121 \,.\, 15 :: 1 \,.\, 9 \,.\, 2 : \text{the discount;} \end{array}$$

and when *either* of these quantities is found the *other* may be had immediately by subtraction.

155. Hence, it appears that the Present Value may be *defined* to be that sum which being *now* put out to interest will amount to the sum *due* at the end of the time.

Since the *Amount* of £100. is the first term for finding the *Discount* and £100. is that for finding the *Interest*, the other terms of the proportions being the same, the Discount of a sum is *less* than its Interest.

It is moreover evident that the discount of a given sum at a given rate of *interest* is *not* proportional to the time: but that the discount is proportional to the sum when the time and rate of interest are given.

In the discharge of a Tradesman's bill, it is *customary* to make a deduction at the *rate* of 5 per cent. *discount* on the sum put down as due at some future time: this *Allowance* being the interest of the *Debt* instead of that of its *Present Value* will be to the payer's advantage, and differs essentially in *principle* from the *Discount* we have just been considering.

Bankers, in *discounting* a bill or promissory note, are in the habit of charging *interest* at 5 per cent. from the day the bill is discounted to the time when the three *days of grace*, allowed by law, have expired.

A distinction should be made between the *Rate of Discount* and the *Rate of Interest*, as is seen immediately in the following familiar instance.

"Five volumes of a work can be bought for a certain sum payable at the end of a year, and six volumes of the same work can be bought for the same sum in ready money: what are the rates of discount and interest?"

Since 1 volume is *discounted* from six, we have

$$\begin{array}{l} \qquad\quad £. \quad\; £. \\ 6 : 1 :: 100 : 16\tfrac{2}{3} = \text{the rate of discount:} \end{array}$$

and because 1 volume is *gained* upon five, we have

$$\begin{array}{l} \qquad\quad £. \quad\; £. \\ 5 : 1 :: 100 : 20 = \text{the rate of interest.} \end{array}$$

156. When discount is reckoned at *Compound Interest* per annum, the same rules may be applied for any number of *entire* years, as appears from Article (145).

Ex. Find the present worth and discount of £2112. 10*s*. due at the end of 2 years, at compound interest of 4 per cent. per annum.

£.	£. s.	£.	£. s.
104 :	2112 . 10 ::	100 :	2031 . 5, which is the sum

due 1 year hence:

104 : 2031 . 5 :: 100 : 1953 . 2 . 6, which is the sum due *now* or the present worth:

and the discount will therefore be

£2112 . 10*s*. – £1953. 2*s*. 6*d*. = £159. 7*s*. 6*d*.

Examples for Practice.

(1) What is the present worth of £157. 10*s*. due 1 year hence, at 5 per cent. simple interest?

Answer: £150.

(2) Required the discount of £355. 5*s*. payable at the end of 4 months, at $4\frac{1}{2}$ per cent. simple interest.

Answer: £5. 5*s*.

(3) Find the discount of £283. 0*s*. 5*d*. for 7 months, at 5 per cent. simple interest.

Answer: £8. 0*s*. 5*d*.

(4) Determine the discount due upon £690. 3*s*. 9*d*. for 9 months, at 3 per cent. simple interest.

Answer: £15. 3*s*. 9*d*.

(5) Find the discount of £298. 0*s*. 10*d*. for 11 months, at 4 per cent. simple interest.

Answer: £10. 10*s*. 10*d*.

(6) Required the present worth of £370. 4*s*. $8\frac{1}{4}$*d*. due 15 months hence, at $4\frac{5}{8}$ per cent. simple interest.

Answer: £350.

(7) What sum will amount to £275. 6*s*. 8*d*. in $1\frac{1}{2}$ years, allowing $4\frac{1}{2}$ per cent. simple interest?

Answer: £257. 18*s*. $5\frac{1}{2}$*d*. $\frac{84}{81}$*f*.

(8) Required the present worth of £241. 12*s*. 4*d*. due at the end of 146 days, at $4\frac{1}{3}$ per cent.

Answer: £237. 10*s*.

(9) How much stock at $92\frac{2}{3}$ must be sold out to pay a bill of £715. 17*s*. due 9 months hence, at 4 per cent. simple interest?

Answer: £750.

(10) A tradesman marks his goods with two prices, one for ready money and the other for credit of six months; what fixed ratio ought the two prices to bear to each other, allowing 5 per cent. per annum, simple interest?

Answer: 40 : 41.

(11) If £2652. 5*s*. be due 3 years hence, what sum will be due at the end of 1 year, if compound interest be allowed at 3 per cent?

Answer: £2500.

(12) What is the present worth of £9724. 1*s*. due 4 years hence, at 5 per cent. compound interest?

Answer: £8000.

V. THE EQUATION OF PAYMENTS.

157. Def. The *Equation of Payments* is the finding of a proper time at which two or more debts due at *different* times should be discharged at *one* payment: and it is here *assumed* that the interests of *all* the debts for their respective periods are together equal to the interest of their *sum* for the *Equated Time*.

Ex. If £100. be due in 3 months, £210. in 2 months and £160. in 5 months, find the equated time.

What is assumed in the definition gives the following equality, since the interests are proportional to the sums and times *jointly*, the rate being supposed the *same* and therefore of no importance in the calculation:

$(100 \times 3) + (210 \times 2) + (160 \times 5) = (100 + 210 + 160) \times$ the equated time: whence, we have

$$\text{the equated time} = \frac{1520}{470} = 3\tfrac{11}{47} \text{ months}:$$

and thence the following Rule is obtained.

Rule. Divide the sum of the products of each payment by its time by the sum of all the payments, and the quotient will be the equated time.

The assumption made in the definition, which implies that the *interest* of the debts payable *before* the equated time from their times to the equated time, should be equal to the *interest* of the debts payable *after* that time from the equated time to their respective times, is not founded in *equity*; because it is evident that by paying a debt before it is due the debtor is entitled to the *discount* only, and that he virtually loses the *interest* which would have accrued from the debt remaining in his hands after its period has expired.

We have seen in Article (155) that interest is greater than discount, and consequently the rule above laid down is in favour of the payer, since a greater allowance is made him than he is really entitled to: but when great nicety is not required, the equated time thus found will not be far from the truth: the *correct* time, however, cannot be obtained without having recourse to *other* than arithmetical principles.

VI. THE RULE OF FELLOWSHIP.

158. Def. *Fellowship* is the rule by means of which two or more persons having a *Joint Stock* or *Common Interest* in a property, are enabled to determine their respective shares of it or of its produce, under the same or different circumstances.

Ex. 1. Two persons form a joint stock by subscribing £3500. and £5000. respectively, and in a certain time they clear £1000.: how must this sum be divided between them?

Here, it is manifest that the share of each person must have the same ratio to the whole gain that his subscription has to the whole stock formed; or, that the *whole* cause must be to each *partial* cause as the *whole* effect is to each *partial* effect:

now, £3500 + £5000 = £8500 is the whole cause,
and £3500 and £5000 are the partial causes:
also, £1000 is the whole effect and the partial effects are the required shares: whence, we have

$$\pounds 8500 : \pounds 3500 :: \pounds 1000 : \text{the first share};$$

$$\text{or, the first share} = \frac{3500 \times 1000}{8500} = \overset{\pounds.}{411} \,.\, \overset{s.}{15} \,.\, \overset{d.}{3\tfrac{1}{2}} \,.\, \overset{f.}{\tfrac{2}{17}}:$$

$$\pounds 8500 : \pounds 5000 :: \pounds 1000 : \text{the second share};$$

$$\text{or, the second share} = \frac{5000 \times 1000}{8500} = \overset{\pounds.}{588} \,.\, \overset{s.}{4} \,.\, \overset{d.}{8\tfrac{1}{4}} \,.\, \overset{f.}{\tfrac{15}{17}}:$$

and these sums make up the £1000. gained.

Here, the ratio of the shares depending *solely* upon the amounts subscribed, the example is termed an instance of *Single* Fellowship.

Ex. 2. A field of grass is rented by two persons for £27.: the former keeps in it 15 oxen for 10 days and the latter 21 oxen for 7 days; find the rent paid by each.

Here, the portions of the rent must evidently be as the numbers of oxen and the numbers of days *jointly*: also, the partial causes are

$$15 \times 10 = 150 \text{ and } 21 \times 7 = 147:$$

and therefore the whole cause is 150 + 147 or 297; whence,

$$297 : 150 :: \overset{\pounds.}{27} : \overset{\pounds.}{13} \,.\, \overset{s.}{12} \,.\, \overset{d.}{8\tfrac{1}{2}} \,.\, \overset{f.}{\tfrac{10}{11}}, \text{ the 1st portion}:$$

$$296 : 147 :: 27 : 13 \,.\, 7 \,.\, 3\tfrac{1}{4} \,.\, \tfrac{1}{11}, \text{ the 2nd portion}:$$

and the sum of both portions is £27. as it ought to be.

This is an instance of *Double* Fellowship, the portions of rent depending upon *two* particulars, the number of oxen put in and the number of days they are kept there.

159. The principles of these examples being independent of the *number* of interests concerned enable us to lay down the following Rule.

Rule. Find the values of the partial causes and also their sum: then, as this sum is to each part of it, so is the whole effect to its corresponding part.

In this rule it is understood that every agent is employed under exactly the *same circumstances:* as, for instance, in the last example each of the oxen is supposed

to consume the *same* quantity of grass the pasturage being *uniform* throughout: but whenever their *relative* qualities are assigned, it will easily be seen that *similar* methods must be pursued.

Ex. If £100. be distributed among 6 men, 9 women and 12 children; what will be received by them, when the shares of a man, a woman and a child are as the numbers 3, 2, 1?

Here, $\left.\begin{aligned} 6 \times 3 &= 18 \\ 9 \times 2 &= 18 \\ 12 \times 1 &= 12 \end{aligned}\right\}$ are the partial causes:

and 48 is the whole cause:

		£.	£. s.	
whence,	48 : 18 ::	100 :	37 . 10,	by the men:
	48 : 18 ::	100 :	37 . 10,	by the women:
	48 : 12 ::	100 :	25 . 0,	by the children.

VII. THE RULE OF ALLIGATION.

160. Def. *Alligation* sometimes called *Alligation Medial* is the rule by means of which the rate or quality of a composition or mixture is found from the rates or qualities of the ingredients of which it is made up.

Ex. If 12 bushels of wheat at 6*s*. a bushel and 15 bushels at 7*s*. a bushel, be mixed together, what will be the value of a bushel of the mixture?

Here, from the most obvious principles, we have

$\left.\begin{aligned} 12 \times 6 &= 72 \\ 15 \times 7 &= 105 \end{aligned}\right\}$ the values of the ingredients:

therefore 72 + 105 = 177*s*. is the value of the mixture which contains 12 + 15 = 27 bushels: whence,

bush. bush. £. s. d. f.
27 : 1 :: 177 : 6 . $6\frac{1}{2}$. $\frac{2}{3}$, the price of a bushel.

The usual *form* of the operation is as follows:

$$6 \times 12 = 72$$
$$7 \times 15 = 105$$

s. d. f.
27) 177 (6 . $6\frac{1}{2}$. $\frac{2}{3}$:

and the number of ingredients being any whatever, we have the following Rule.

RULE. Divide the sum of the products of the ingredients and their respective rates by the sum of the ingredients, and the quotient will be the rate of the mixture.

Examples for Practice.

(1) If £75. be due in 4 months, £125. in 5 months and £150. in 7 months: what is the equated time?

Answer: $5\frac{9}{14}$ months.

(2) What will be the equated time of payment of £200. due at 3 months, £300. at 8 months and £500. at 12 months?

Answer: 9 months.

(3) Find the equated time of payment, when $\frac{1}{2}$ of a sum of money is due at 3 months, $\frac{1}{5}$ at 8 months and the remainder at 15 months.

Answer: $7\frac{3}{5}$ months.

(4) *A* owed *B* £750. to be paid in 15 months, but at 12 months he paid him £250.: at what time was the remainder due?

Answer: $16\frac{1}{2}$ months.

(5) Divide £1000. among three persons, so that their shares shall be as the numbers 2, 5, 9.

Answer: £125., £312. 10*s*. and £562. 10*s*.

(6) Of £2180, *A*'s share is to *B*'s share as 2 to 3, *B*'s is to *C*'s as 4 to 7 and *C*'s is to *D*'s as 5 to 11; find the share of each.

Answer: *A*'s is £200., *B*'s is £300., *C*'s is £525. and *D*'s is £1155.

(7) Three partners put into business the sums of £300., £400. and £500., and at the end of a certain time they gained £600.: find the share of each.

Answer: £150., £200. and £250.

(8) Three persons forming a joint stock of £45000., gain by trading £15000.; and of this their shares are

£7500., £5000. and £2500.: find the portion of stock contributed by each.

Answer: £22500., £15000. and £7500.

(9) A person bequeathed by will the following legacies: £1500. to *A*, £875. to *B*, £525. to *C* and £350. to *D*: but when his property was sold it produced only £2437. 10*s*.: how much did he really leave to each?

Answer: £1125. to *A*, £656. 5*s*. to *B*, £393. 15*s*. to *C* and £262. 10*s*. to *D*.

(10) If *A* advance £1500. for 9 months and *B* £1200. for 6 months: what share of a gain of £1150. belongs to each?

Answer: £750. to *A* and £400 to *B*.

(11) If *A* contribute £6000. for 5 months, *B* £5000. for 6 months, *C* £4000. for $7\frac{1}{2}$ months and *D* £2500. for 12 months, in the formation of a joint stock: divide a profit of £4760. equitably among them.

Answer: The share of each is £1190.

(12) Three merchants *A*, *B*, *C* engage in commerce; *A* with £1000. for 12 months, *B* with £1800. for 7 months and *C* with £2500. for 4 months, and they gain £350.: what share of the gain belongs to each?

Answer: £121. 7*s*. $8\frac{3}{4}d.\ \frac{137}{173}f.$ to *A*, £127. 9*s*. $1\frac{1}{2}d.\ \frac{66}{173}f.$ to *B* and. £101. 3*s*. $1\frac{1}{4}d.\ \frac{143}{173}f.$ to *C*.

(13) Three persons with a joint stock gain £3650.: the first advances $\frac{1}{3}$ of the capital for $\frac{1}{4}$ of the time, the second $\frac{1}{4}$ of the capital for $\frac{1}{2}$ of the time and the third the remainder of the capital for the whole time: find their shares.

Answer: £486. 13*s*. 4*d*., £730. and £2433. 6*s*. 8*d*.

(14) A prize of £3825. is to be divided among 3 officers, 12 assistants and 100 men, in proportion to their pay and time of service jointly: the officers have £5. a month and have served 9 months: the assistants who have £2. 10*s*. a month have served 6 months and the men have served 3 months at £1. 10*s*. a month. What is the share of each?

Answer: £225. of each officer, £75. of each assistant and £22. 10*s*. of each of the men.

(15) A wine merchant mixes 20 gallons of wine at 12*s*. a gallon, 25 gallons at 14*s*. and 36 gallons at 16*s*.: what will be the price of a gallon of the mixture?

Answer: 14*s*. $4\frac{1}{2}$*d*. $\frac{20}{27}$*f*.

(16) A mixture is made of 10 bushels of flour at 3*s*. 8*d*., 21 bushels at 3*s*. 10*d*. and 35 bushels at 4*s*.: what is the price of a bushel of it?

Answer: 3*s*. $10\frac{3}{4}$*d*. $\frac{1}{33}$*f*.

VIII. THE DOCTRINE OF EXCHANGES.

161. Def. 1. *Exchange* is the rule by means of which it is ascertained what sum of money of one country is equivalent to a *given* sum of another, according to some *settled* rate of commutation: and the operations necessary to calculate this must, from the nature of the case, be applications of the Rule of Proportion.

The *Course of Exchange* is used to express the sum of money of any place given in exchange for a *fixed* sum of that of another: and the *Par of Exchange* denotes the sum of money of any place, which is of the same *intrinsic* value as that fixed sum.

Ex. How many pounds Flemish can I receive for £1050. sterling, the course of exchange being 35 shillings Flemish for £1. sterling?

Here, from the nature of the question, we have

```
£.     £.       s. f.
1 : 1 0 5 0 :: 3 5

       5 2 5 0
     3 1 5 0

 2, 0) 3 6 7 5,0 shillings Flemish:

     £ 1 8 3 7 . 1 0 the sum Flemish required.
```

In questions of this kind, all that is necessary to be known is the course of exchange and the subdivisions of the monies to be commuted.

162. Def. 2. The *Arbitration* or *Comparison* of Exchanges is the determining what rate of exchange,

called the *Par of Arbitration,* between any number of places corresponds with, or is equivalent to, any assigned rates between each of them and another place: and a knowledge of this subject will enable a person to judge how he may remit his money from one place to another with the greatest advantage.

Arbitration is styled *simple* or *compound,* according as *three* or *more* places are concerned.

Ex. If the exchange between *Amsterdam* and *Paris* be 54*d.* for 1 crown, and between *Amsterdam* and *London* be 33*s.* 9*d.* for £1.; what is the par of exchange or the arbitrated price between *Paris* and *London?*

Here, 1 crown at *Paris* = 54 pence at *Amsterdam:*

240 pence in *London* = 405 pence at *Amsterdam:*

thus, we obtain the equality of ratios,

$$\frac{1 \text{ crown in } Paris}{240 \text{ pence in } London} = \frac{54}{405} = \frac{2}{15};$$

whence, 1 crown at $Paris = \frac{2}{15} \times 240d. = 32d.$ in *London:*

that is, 32*d.* per crown is the arbitrated price between *London* and *Paris.*

If we arrange the equalities so that the first term of one shall always be of the *same kind* as the second of that which immediately precedes it, as follows:

1 crown at *Paris* = 54 pence at *Amsterdam,*

405 pence at *Amsterdam* = 240 pence in *London,*

and multiply together the corresponding terms retaining the *names* only of the first and last countries and their *denominations* of money, we shall have

405 crowns at *Paris* = 54 × 240 pence in *London:*

and therefore 1 crown at $Paris = \frac{54 \times 240}{405} = \overset{d.}{32}$ in *London,*

as before: and a proceeding of this kind is distinguished by the name of the *Chain Rule,* from the *connection* of the *first* and *last* terms being ascertained through those which are *intermediate.*

Examples for Practice.

(1) How much English money is equivalent to 1785 francs 6 decimes, at 24 francs per pound sterling?

Answer: £74. 8*s.*

(2) Reduce £156. 15*s.* to francs, the exchange being at 23.5 francs per pound sterling.

Answer: 3683.625 francs.

(3) If £100. be due from *London* to *Paris* when £1. is worth 25 francs: what sum must be remitted when a guinea is exchanged for 27 francs?

Answer: £97. 4*s.* $5\frac{1}{3}d.$

(4) If the course of exchange between *London* and *Amsterdam* be 33*s.* 6*d.* Flemish per pound sterling and between *London* and *Lisbon* be 50*d.* sterling per milree: find the arbitrated rate of exchange between *Amsterdam* and *Lisbon*.

Answer: $83\frac{3}{4}d.$ Flemish per milree.

(5) A person in *London* owes another at *Petersburg* 500 rubles, exchange at 40*d.* sterling per ruble: but remits to *Paris* at 24 francs per pound sterling; thence to *Lisbon* at 500 rees for 3 francs; thence to *Amsterdam* at 20 stivers per crusado of 400 rees and thence to *Petersburg* at 25 stivers per ruble: find the arbitrated rate between *London* and *Petersburg* and the gain or loss by the circuitous mode of remittance.

Answers: 30*d.* per ruble and the gain is £20. 16*s.* 8*d.*

(6) The rates of exchange being £1. = 25.4 francs, 3.75 francs = 105 kreutzen, 60 kreutzen = 1 florin and the cost of travelling in *Germany* being $1\frac{1}{2}$ florins per German mile which is equal to $4\frac{1}{2}$ English miles: find the expense, in English money, of travelling 381 English miles in Germany.

Answer: £10. 14*s.* $3\frac{3}{7}d.$

163. The *Course of Exchange* between two countries fluctuates according to circnmstances which cannot be entered into here; but the *lower* the course of exchange, the *more favourable* is it to the country in whose money it is estimated, and *vice versâ.*

Thus, between *London* and *Amsterdam*, when the course of exchange is 9 guilders per pound sterling, it will evidently require *more* sterling money to pay a debt in *Amsterdam* and *fewer* guilders to discharge one in *London*, than if the course of exchange were 11 guilders: for, the merchant of *Amsterdam* has to *buy* pounds sterling to remit to *London* and the *London* merchant has to *sell* pounds sterling in order to purchase guilders for a remittance to *Amsterdam*. The exchanges may therefore be considered favourable to *this country*, when the courses of exchange run high in *foreign countries* with which it trades, and *vice versâ*.

The reader is referred to the last Edition of Dr. KELLY'S *Universal Cambist* for *practical* information on this subject.

MISCELLANEOUS QUESTIONS.

164. In this section are presented a few miscellaneous Questions which could not with propriety be arranged under any of the preceding heads and are still of too much importance to be passed over without notice, in a work like the present.

QU. 1. How many dozens of wine at £2. a dozen must be given in exchange for 27 yards of broad cloth at 32*s*. a yard?

The price of the cloth is $27 \times 32 = 864$*s*.:

whence, 40*s*. : 864*s*. :: 1 doz. : $21\frac{3}{5}$doz.;

that is, $21\frac{3}{5}$ dozens of wine are of equal value with 27 yards of cloth.

Questions of this kind are termed instances of *Barter* or *Truck*.

QU. 2. If a grocer by selling tea at 6*s*. 6*d*. a pound clear one-sixth of the money: what will he clear per cent. by selling it at 7*s*. a pound?

Here, $\frac{1}{6}$ of 6*s*. 6*d*. = 1*s*. 1*d*.; whence 5*s*. 5*d*. is the price per lb. the tea cost him: therefore

5*s*. 5*d*. : 7*s*. :: £100. : £129. 4*s*. $7\frac{1}{4}$*d*. $\frac{7}{13}$*f*.;

and £129. 4*s*. $7\frac{1}{4}$*d*. $\frac{7}{13}$*f*. is the *increased* value of £100. at this rate: that is, the *gain* per cent. is £29. 4*s*. $7\frac{1}{4}$*d*. $\frac{7}{13}$*f*.

Qu. 3. A person loses at the rate of 10 per cent. by selling cloth at 15*s.* a yard: how ought it to have been sold to gain 20 per cent.?

Since he loses $\frac{1}{10}$th part, he receives only 9 parts out of 10 or 90 parts out of 100: whence,

$$\overset{\pounds.}{90} : \overset{\pounds.}{100} :: \overset{s.}{15} : \overset{s.}{16} . \overset{d.}{8},$$

the *prime cost* of 1 yard: for the same reason, we have

$$\overset{\pounds.}{100} : \overset{\pounds.}{120} :: \overset{s.}{16} . \overset{d.}{8} : \overset{s.}{20},$$

or £1. is the price per yard, in order to realize a *profit* of 20 per cent.

Questions of this description are classed under the heads, *Profit and Loss, Loss and Gain* and *Per-centage.*

Qu. 4. Required the neat weight of 27 cwt. 1 qr. 14lbs., tare being allowed at the rate of 16lbs. per cwt.

Here, by the rules of Practice, we have

$$\begin{array}{c|c|l} 16 & \frac{1}{7} & 27 \;.\; \overset{qr.}{1} \;.\; 14 = \text{gross:} \\ & & \;\;3 \;.\; 3 \;.\; 18 = \text{tare:} \\ \hline & & 23 \;.\; 1 \;.\; 24 = \text{neat weight.} \end{array}$$

Questions of this nature are usually inserted under a Rule called *Tare and Tret,* which comprises all allowances made upon goods on any ground whatever, whether by *custom* or by *special agreement.*

Qu. 5. If two men *A* and *B* together can perform a piece of work in 10 days and *A* by himself can do it in 18 days: what time will it take *B* to do it?

Assuming 1 to represent the piece of work, we have

$$\frac{1}{18} = \text{work done by } A \text{ in } 1 \text{ day:}$$

$$\frac{10}{18} = \frac{5}{9} = \ldots\ldots\ldots\ldots\ldots\ldots A \text{ in } 10 \text{ days:}$$

$$\text{hence, } 1 - \frac{5}{9} = \frac{4}{9} = \ldots\ldots\ldots\ldots\ldots\ldots B \text{ in } 10 \text{ days:}$$

$$\text{wherefore } \frac{4}{9} : 1 :: 10\text{days} : 22\tfrac{1}{2} \text{ days;}$$

or, *B* can do the work in $22\frac{1}{2}$days.

Qu. 6. Three agents A, B, C can produce a given effect in 12 hours; also, A and B can produce it in 16 hours and A and C in 18 hours: in what time can each of them produce it separately?

Here, $\frac{1}{16}$ = effect produced by A and B in 1 hour:

$\frac{12}{16} = \frac{3}{4} = \ldots\ldots\ldots\ldots\ldots\ldots$ A and B in 12 hours:

whence, $1 - \frac{3}{4} = \frac{1}{4} = \ldots\ldots\ldots\ldots$ C in 12 hours:

$$\text{and therefore } \frac{1}{4} : 1 :: 12 \text{ hours} : 48 \text{ hours},$$

the time in which C alone can produce it:

again, $\frac{1}{18}$ = effect produced by A and C in 1 hour:

$\frac{12}{18} = \frac{2}{3} = \ldots\ldots\ldots\ldots\ldots\ldots$ A and C in 12 hours:

and $1 - \frac{2}{3} = \frac{1}{3} = \ldots\ldots\ldots\ldots\ldots\ldots$ B in 12 hours:

$$\text{whence, } \frac{1}{3} : 1 :: 12 \text{hrs.} : 36 \text{hours},$$

the time in which B alone can produce it:

also, $\frac{1}{12}$ = effect produced by A, B and C in 1 hour:

and $\frac{1}{36} + \frac{1}{48} = \frac{7}{144} = \ldots\ldots\ldots\ldots\ldots\ldots$ B and C in 1 hour:

whence, $\frac{1}{12} - \frac{7}{144} = \frac{5}{144} = \ldots\ldots$ A in 1 hour:

$$\text{therefore } \frac{5}{144} : 1 :: 1 \text{hr.} : 28\tfrac{4}{5} \text{hrs.},$$

the time in which A can produce the effect proposed.

Qu. 7. Distribute £200. among A, B, C and D, so that B may receive as much as A, C as much as A and B together and D as much as A, B and C together.

If the share of A be represented by 1, then
the share of B will be represented by 1:
the share of C by $1 + 1 = 2$:
and the share of D by $1 + 1 + 2 = 4$:

whence, the question is merely to divide £200. into four parts having the same relations as the numbers 1, 1, 2, 4:

$$\text{also,}\quad 1 + 1 + 2 + 4 = 8,$$

and the Rule of Fellowship gives the following proportions:

			£.		£.	
8	:	1 ::	200	:	25,	the share of A;
8	:	1 ::	200	:	25,	the share of B;
8	:	2 ::	200	:	50,	the share of C:
8	:	4 ::	200	:	100,	the share of D.

The same mode of reasoning will be applicable whatever be the number of persons concerned.

Qu. 8. At what times between 2 and 3 o'clock, are the hour and minute hands of a clock together, at right angles and in opposite directions?

At two o'clock, the hour hand is *two* of the portions called *hours* of one hand and *five minutes* of the other in advance of the minute hand; and their rates being as $1 : 12$, the minute hand *gains* 55 in 60 or 11 in 12 upon the hour hand: whence we have

$$11 : 12 :: 2 : 2\tfrac{2}{11},$$

the *hours* when the minute and hour hands are *together*.

Again, when they are at right angles, the minute hand must have gained $2 + 3 = 5$ portions, and we have

$$11 : 12 :: 5 : 5\tfrac{5}{11};$$

and therefore at $5\tfrac{5}{11} \times 5$ or $27\tfrac{3}{11}$ *minutes* past two, the hands are at *right angles*.

Also, if they point in opposite directions, $2 + 6 = 8$ portions must be gained by the minute hand, and therefore we have

$$11 : 12 :: 8 : 8\tfrac{8}{11},$$

or, the hands will be in *opposite directions* at $8\tfrac{8}{11} \times 5$ or $43\tfrac{7}{11}$ *minutes* past two.

When the minute hand has gained $2+9=11$ portions, the two hands will be at right angles again, and

$$11 : 12 :: 11 : 12,$$

which shews that this circumstance occurs at 60 *minutes* past two or at three o'clock, as we know to be the case.

Qu. 9. Two clocks point out 12 at the same instant; one of them gains 7″ and the other loses 8″ in 12 hours: after what interval will one have gained half an hour of the other and what o'clock will each then shew?

Here, $7''+8''=15''$ is the separation which takes place in 12 hours, and $\frac{1}{2}$ hour $=30'=1800''$: whence,

$$15'' : 1800'' :: 12\text{hrs.} : 1440\text{hrs.};$$

that is, in 1440 hours or 60 days they will be separated 30 minutes or half an hour.

Also, the first gains 7″ in 12 hours or 14″ in 1 day:

$$\text{and } 1 \text{ day} : 60 \text{days} :: 14'' : 14';$$

and therefore it will shew 12 hours 14 minutes.

The second loses 8″ in 12 hours, or 16″ in 1 day;

$$\text{and } 1 \text{ day} : 60 \text{ days} :: 16'' : 16';$$

whence the time pointed out by it will be 12 hrs. −16 min. or 11 hours 44 minutes: and it will be observed that the times differ by half an hour, as they ought.

Examples for Practice.

(1) How much cloth at 14*s.* 6*d.* a yard, must be given for 3cwt. 3qrs. of sugar at £3. 4*s.* per cwt.?

Answer: 16yds. $2\frac{6}{29}$qrs.

(2) If 126 yards of cloth be bartered for 3hhds. of brandy at 6*s.* 8*d.* per gallon, what is the price of the cloth per yard?

Answer: 10*s.*

(3) If I buy goods at £3. 16*s.* 8*d.* per cwt.: how must I retail them per lb. to gain 15 per cent.?

Answer: $9\frac{1}{4}d.\ \frac{11}{14}f.$

(4) If by selling tea at 6*s.* 4*d.* per lb. a grocer lose 6 per cent.; what did it cost him per lb.?

Answer: $6s.\ 8\frac{3}{4}d.\ \frac{19}{47}f.$

(5) A grocer bought 2tons. 3cwt. 3qrs. of sugar for £120. and paid £2. 10*s*. for expenses: what must he sell it at per cwt. to clear 50 per cent.?

Answer: £4. 4*s*.

(6) A person by disposing of goods for £182. loses at the rate of 9 per cent.: what ought they to have been sold for to realize a profit of 7 per cent.?

Answer: £214.

(7) A person buys 400 yards of silk for £80. and sells 300 yards at 5*s*. 6*d*. a yard, and the rest which is damaged at 2*s*. a yard: what does he gain or lose per cent.?

Answer: He gains £15. 12*s*. 6*d*. per cent.

(8) A stationer sold quills at 11*s*. a thousand, by which he cleared $\frac{3}{8}$ of the money and he afterwards raised them to 13*s*. 6*d*. a thousand: what did he gain per cent. by the latter price?

Answer: £96. 7*s*. $3\frac{1}{4}$*d*. $\frac{1}{11}$*f*.

(9) At what price must a commodity purchased at £14. 5*s*. per cwt. be sold to gain 21 per cent.; and what quantity of it must be sold at that rate to clear £100.?

Answers: £17. 4*s*. $10\frac{1}{5}$*d*. per cwt. and the quantity is 33cwt. 1qr. 18lbs. $11\frac{7}{171}$oz.

(10) A merchant bought 160 quarters of wheat at 41*s*. 3*d*. per quarter and sold it at 58*s*. 4*d*.: what was his gain? At what price ought it to have been sold to gain £33.?

Answers: £136. 13*s*. 4*d*. and 45*s*. $4\frac{1}{2}$*d*.

(11) The prime cost of a cask of wine of 38 gallons is £25. and 8 gallons are lost by leakage: at what price per gallon must the remainder be sold so as to gain 10 per cent. upon the prime cost?

Answer: 18*s*. 4*d*.

(12) Divide £64. among *A*, *B* and *C*, so that *A* may have three times as much as *B* and *C* may have one third of what *A* and *B* have together.

Answer: *A* has £36., *B* has £12. and *C* has £16.

(13) A person paid a tax of 10 per cent. upon his income: what must his income have been when after he had paid the tax, there was £1250. remaining?

Answer: £1388. 17*s*. $9\frac{1}{4}$*d*. $\frac{1}{3}$*f*.

(14) A grocer had 150lbs. of tea of which he sold 50lbs. at 9*s*. per lb. and found that he was thereby gaining $7\frac{1}{2}$ per cent.; at what rate must he sell the remaining 100lbs., so as to clear 10 per cent. upon the whole?

Answer: 9*s*. $3\frac{3}{4}$*d*. $\frac{8}{43}$*f*.

(15) A mixture of wine and water of 32 measures contains one measure of wine: how much water must be added to this mixture that 32 measures of it may contain $\frac{1}{8}$ of a measure of wine?

Answer: 224 measures.

(16) A hare starts 40 yards before a greyhound and is not perceived by him till she has been up 40 seconds: she gets away at the rate of 10 miles an hour and the dog pursues her at the rate of 18 miles an hour: how long will the course last and what distance will the hare have run?

Answers: $60\frac{5}{22}$ seconds and 490 yards.

(17) At what time between twelve and one o'clock, do the hour and minute hands of a watch point in directions exactly opposite?

Answer: 32min. $43\frac{7}{11}$ sec. past 12.

(18) A church clock is set at 12 o'clock on Saturday night; at noon on Tuesday it is 3 minutes too fast: supposing the rate regular find the true time when the clock strikes four on Thursday afternoon.

Answer: $5\frac{715}{1201}$ minutes before four.

(19) If 5 men or 7 women can perform a piece of work in 35 days: in what time can 7 men and 5 women do the same?

Answer: $16\frac{41}{74}$ days.

(20) A certain number of men mow 4 acres of grass in 3 hours and a certain number of others mow

8 acres in 5 hours: how long will they be in mowing 11 acres if they all work together?

Answer: $3\frac{3}{4}$ hours.

(21) If 15 men, 12 women and 9 boys can complete a piece of work in 50 days; what time would 9 men, 15 women and 18 boys take to do twice as much, the parts done by each in the same time being as the numbers 3, 2 and 1?

Answer: 104 days.

(22) If *A* can do a piece of work in 5 days, *B* twice as much in 7 days and *C* four times as much in 11 days: in what time can *A*, *B* and *C* together do three times the said work?

Answer: 3 days 12hrs. $46\frac{26}{109}$min.

(23) If *A* and *B* together can build a hut in 18 days and with the assistance of *C* they can do it in 11 days; in what time can *C* do it by himself?

Answer: $28\frac{2}{7}$ days.

(24) If *A* can do a piece of work in 1 hour, *B* in 3 hours, *C* in 5 hours and *D* in 7 hours; in what time can they do eleven times as much all working together at their respective rates?

Answer: 6hrs. 33min. 45sec.

(25) *A* and *B* can do a piece of work in 10 days, *A* and *C* in 12 days and *B* and *C* in 14 days: in what times can they do it jointly and separately?

Answers: All together in $7\frac{91}{107}$ days, *A* in $17\frac{41}{47}$ days, *B* in $22\frac{26}{37}$ days and *C* in $36\frac{12}{23}$ days.

(26) If *A*, *B* and *C* could reap a field in 18 days, *B*, *C* and *D* in 20 days, *C*, *D* and *A* in 24 days and *D*, *A* and *B* in 27 days: in what times would it be reaped by them altogether, and by each of them separately?

Answers: By them altogether in $16\frac{56}{199}$ days, by *A* in $87\frac{21}{37}$ days, by *B* in $50\frac{5}{8}$ days, by *C* in $41\frac{1}{79}$ days and by *D* in $170\frac{10}{19}$ days.

CHAPTER VII.

INVOLUTION AND EVOLUTION,

WITH THE ARITHMETIC OF SURDS.

INVOLUTION.

165. Def. A *Power* of a number is the number which arises from successive multiplications by itself: the operation by which it is obtained is termed *Involution*; and the *Degree* or *Order* of the power is denoted by the *number* of factors employed.

Thus, taking the number 2, we shall have the powers of it as follows:

$2 = 2$, the first power of 2:

$2 \times 2 = 4$, the second power of 2:

$2 \times 2 \times 2 = 8$, the third power of 2:

$2 \times 2 \times 2 \times 2 = 16$, the fourth power of 2:

$2 \times 2 \times 2 \times 2 \times 2 = 32$, the fifth power of 2:

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$, the sixth power of 2:

and so on, as far as we please:

but instead of expressing these multiplications at *length*, which would soon become inconvenient, we denote the same operations by means of *Indices* or small figures placed a little above the line to the right of the quantities whose powers are intended to be exhibited: thus, what is put down above may be denoted by

$$2^1 = 2: \qquad 2^3 = 8: \qquad 2^5 = 32:$$
$$2^2 = 4: \qquad 2^4 = 16: \qquad 2^6 = 64: \ \&c.:$$

where the *Index* sometimes called the *Exponent* is *equal* to the number of *factors* and is greater by *one* than the number of *operations*.

In the same manner the *second* powers of the *nine* digits are expressed: thus,

$1^2=1$:	$4^2=16$:	$7^2=49$:
$2^2=4$:	$5^2=25$:	$8^2=64$:
$3^2=9$:	$6^2=36$:	$9^2=81$:

and their *third* powers will be written as follows:

$1^3=1$:	$4^3=64$:	$7^3=343$:
$2^3=8$:	$5^3=125$:	$8^3=512$:
$3^3=27$:	$6^3=216$:	$9^3=729$.

The *second* and *third* powers of numbers are styled their *Squares* and *Cubes* in reference to their application to *Geometry*, as will be seen hereafter: and the operations by which *all* powers are obtained are merely those of Multiplication.

166. To find the powers of a vulgar fraction or of a quantity expressed decimally, a similar process is used: thus,

$\left(\frac{2}{3}\right)^1=\frac{2}{3}$:	$(2.5)^1=2.5$:
$\left(\frac{2}{3}\right)^2=\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$:	$(2.5)^2=2.5\times2.5=6.25$:
$\left(\frac{2}{3}\right)^3=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{8}{27}$:	$(2.5)^3=2.5\times2.5\times2.5=15.625$:
&c.:	&c.:

and exactly in the same manner the powers of a quantity expressed by factors are found:

thus, the square of $2\times7=(2\times7)\times(2\times7)$

$$=2\times2\times7\times7=2^2\times7^2=4\times49=196.$$

Hence it appears that a power of a fraction is equal to the fraction formed by *raising* both its numerator and denominator to the power, and that the power of a quantity formed by factors is found by raising each factor to the power.

A mixed quantity is represented as a simple fraction or as a decimal, before the process is applied.

167. This notation furnishes important conclusions with respect to powers.

Thus, since $3^2 = 3 \times 3$ and $3^4 = 3 \times 3 \times 3 \times 3$, we have

$$3^4 \times 3^2 = (3 \times 3 \times 3 \times 3) \times (3 \times 3)$$
$$= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 3^{4+2}:$$
$$3^4 \div 3^2 = (3 \times 3 \times 3 \times 3) \div (3 \times 3)$$
$$= \frac{3 \times 3 \times 3 \times 3}{3 \times 3} = 3 \times 3 = 3^2 = 3^{4-2}$$

from which we infer that the *Multiplication* and *Division* of powers of the same quantity are *expressed* by the *Addition* and *Subtraction* of their indices.

Similarly, we have the fourth power of 3^2 expressed by $3^2 \times 3^2 \times 3^2 \times 3^2 = 3^8 = 3^{2 \times 4}$; or, the *Involution* of powers is expressed by the *Multiplication* of their indices: and conversely.

Ex. Let it be required to find the 6th power of 13.

$$\text{Here,}\quad 13^6 = 13^{1+2+3} = 13^1 \times 13^2 \times 13^3$$
$$= 13 \times 169 \times 2197 = 4826809:$$

and the same result will be obtained by effecting any of the operations indicated below:

$$13^6 = 13^2 \times 13^4 = 13^3 \times 13^3 = 13^5 \times 13.$$

168. When one power of a quantity is divided by a higher power of the same quantity, the quotient may be expressed by the power of a fraction: thus,

$$7^2 \div 7^4 = (7 \times 7) \div (7 \times 7 \times 7 \times 7)$$
$$= \frac{7 \times 7}{7 \times 7 \times 7 \times 7} = \frac{1}{7 \times 7} = \frac{1}{7^2} = \left(\frac{1}{7}\right)^2.$$

Also, from these Articles we ascertain that

$$7^4 \div 7^2 = 7^{4-2} = 7^2:$$
$$7^2 \div 7^4 = \frac{1}{7^{4-2}} = \frac{1}{7^2}:$$

where the *difference* of the indices is employed in the numerator or denominator according as the dividend or divisor is the higher power.

If the indices of the dividend and divisor be the same, this notation *extended* will give us the representation of unity or 1 in the *form* of the power of any number or quantity whatever, as 7 for instance, whose index is 0,

$$\text{since}\quad 1 = 7^4 \div 7^4 = 7^{4-4} = 7^0.$$

EVOLUTION.

169. **Def.** A *Root* of a number is such a number as being multiplied into itself one or more times produces it; and the operation by which this root is obtained is called *Evolution.*

Thus, the second or *square* root of 16 is 4, because the square of 4 is 16, or $4^2 = 4 \times 4 = 16$.

The third or *cube* root of 512 is 8, since the cube of 8 is 512, or $8^3 = 8 \times 8 \times 8 = 512$:

and similarly of vulgar fractions and decimals.

This operation is expressed by the sign $\sqrt{}$ which is called the *Radical Sign*, with a small figure placed on its left to *particularize* the root intended: thus,

$$\sqrt[2]{16} = 4 \quad \text{and} \quad \sqrt[3]{512} = 8:$$

but the *square* root is denoted by the sign $\sqrt{}$ *only*, without the small figure, as being of most frequent occurrence.

These operations are also indicated by means of the primitive fractions $\frac{1}{2}$, $\frac{1}{3}$, &c., used as *indices*, so that the indices $\frac{1}{2}$, $\frac{1}{3}$, &c., denote operations exactly the reverse of those expressed by the indices 2, 3, &c., respectively: thus,

$$4^2 = 16, \qquad 16^{\frac{1}{2}} = 4;$$
$$8^3 = 512, \qquad 512^{\frac{1}{3}} = 8.$$

EXTRACTION OF THE SQUARE ROOT.

170. In this operation, having only *one* magnitude to work with, we cannot avail ourselves of any of the *fundamental* operations of *Arithmetic:* we shall therefore merely put down such instructions as will enable the student to extract the square root, without entering into the reasons upon which they are founded, these reasons admitting of a much clearer exposition by means of *Algebraical Symbols* than any that could be given in particular numbers. See the *Appendix*.

171. Repeating what was said in Article (165), we have

Digits:

1, 2, 3, 4, 5, 6, 7, 8, 9:

Squares:

1, 4, 9, 16, 25, 36, 49, 64, 81:

whence, by mere inspection, we are enabled to find the square roots of all numbers that can be produced by the *squaring* of a single figure: but it is evident that this statement will not be sufficient for finding the square roots of quantities consisting of more than *two* figures; and recourse must therefore be had to other expedients.

172. *From the number of figures in any proposed quantity, to find the number of figures in its square root.*

Since, the square root of 1 is 1:
the square root of 100 is 10:
the square root of 10000 is 100:
the square root of 1000000 is 1000: &c.:

we see immediately that the square root of a number of fewer than three figures must consist of only one figure: that of a number of more than two figures and fewer than five, of two figures: that of a number of more than four figures and fewer than seven, of three figures, and so on: whence it follows, that if a dot or full point be placed over every alternate figure, beginning at the *units'* place, the number of such points will be the same as the number of figures in the square root.

This is called the *Rule for Pointing*, and may easily be extended to decimals: thus,

since, the square root of .01 is .1:
the square root of .0001 is .01:
the square root of .000001 is .001: &c.:

we infer that the quantity proposed must first be made to have an *even* number of decimal places, and then the pointing must proceed from the place of *units* towards the right hand over every alternate figure as before: and the number of such points will be the same as the number of decimal places in the square root.

Rule for the Extraction of the Square Root.

Point the alternate figures of the number proposed, beginning at the place of units, so as to form as many *periods* of two figures each as possible: find the greatest square number contained in the first period on the left hand, put down its root on the right as in division, and subtract it from that period. To the remainder bring

down the next period for a dividend, double the root just found for a divisor, and find how often it is contained in this dividend exclusive of the figure on its right hand, annex this quotient to the figures in both the quotient and divisor: multiply the divisor thus *completed* by the last figure of the quotient, subtract the product as before, and bring down to the remainder the period which comes next in order: repeat the process till every period in succession is disposed of, and the root or an approximation to it will thus be obtained.

The divisors *tried* as above, or the *trial* divisors, will frequently be taken too large when the dividend consists of only *two* or *three* figures, but not so in other cases: and attention to this circumstance will save trouble.

Ex. 1. Find the square roots of 1444 and 16129.

Proceeding according to the directions given in the Rule, we have

$$\begin{array}{rl} & 1\dot{4}4\dot{4}\,(38 \\ & 9 \\ \hline 68) & 544 \\ & 544 \\ \hline \end{array} \qquad\qquad \begin{array}{rl} & 16129\,(127 \\ & 1 \\ \hline 22) & 61 \\ & 44 \\ \hline 247) & 1729 \\ & 1729 \\ \hline \end{array}$$

or, the square roots of 1444 and 16129 are 38 and 127 respectively: and these operations may easily be verified by squaring the numbers 38 and 127: also, the importance of the remark last made will be apparent.

Ex. 2. Required the square roots of the mixed decimals 22.09 and 104.7931.

$$\begin{array}{rl} & 2\dot{2}.0\dot{9}\,(4.7 \\ & 16 \\ \hline 87) & 609 \\ & 609 \end{array} \qquad\qquad \begin{array}{rl} & \dot{1}0\dot{4}.7\dot{9}3\dot{1}\,(10.23 \\ & 1 \\ \hline 202) & 0479 \\ & 404 \\ \hline 2043) & 7531 \\ & 6129 \\ \hline & 1402 \end{array}$$

The former of these is a *complete* square whose root is 4.7; but the latter is not, its approximate root being 10.23 with a remainder .1402: and it will be found upon trial, that $(10.23)^2 + .1402 = 104.7931$: also, this approximation might evidently have been carried farther, by affixing to the right hand of the quantity proposed, periods of *ciphers* which do not affect its value.

Ex. 3. Determine the square roots of the fractional quantities $\frac{144}{169}$ and $1278\frac{7}{25}$.

From Article (166), we see that the square root of a fraction may be obtained by finding the square roots of its numerator and denominator separately: whence, the square root of $\frac{144}{169} = \frac{\text{the square root of } 144}{\text{the square root of } 169} = \frac{12}{13}$.

Hence also, since $1278\frac{7}{25} = \frac{31957}{25}$, the square root may be found as above: but as the terms are seldom complete squares, it is usual to express the fraction decimally before the rule is applied; and in this instance, we shall have the approximate square root of 1278.28 = 35.753 &c., which might have been extended to more decimal places at pleasure.

Ex. 4. Extract the square root of the recurring decimal $1.\dot{7}$.

Here $1.\dot{7} = \frac{16}{9}$ and therefore the square root is $\frac{4}{3} = 1.\dot{3}$: but it generally happens that the corresponding vulgar fraction is not a complete square, and the approximate root must then be found by the ordinary method, though it will not be a recurring decimal.

It may here be observed, that the remainder at any stage of the operation must not *exceed* twice the corresponding quotient or portion of the root: and when a few figures of the root are obtained, their number may nearly be *doubled* by Division only.

Examples for Practice.

(1) Find the square roots of 676, 21025, 288369 and 998001.

Answers: 26, 145, 537 and 999.

(2) Determine the square roots of 2025, 692224, 33016516 and 45859984.

Answers: 45, 832, 5746 and 6772.

(3) What are the square roots of 5774409, 62805625, 182493081 and 3915380329?

Answers: 2403, 7925, 13509 and 62573.

(4) Required the square roots of 33.64, 1082.41, 22.8484 and 187.4161.

Answers: 5.8, 3.29, 4.78 and 13.69.

(5) Find the square roots of .0064, .005329, .00053361 and .00038025.

Answers: .08, .073, .0231 and .0195.

(6) Extract the square roots of $\frac{4}{25}$, $\frac{169}{256}$, $\frac{841}{1369}$ and $\frac{2304}{3481}$.

Answers: $\frac{2}{5}$, $\frac{13}{16}$, $\frac{29}{37}$ and $\frac{48}{59}$.

(7) What are the square roots of $4\frac{25}{36}$, $10\frac{39}{49}$, $345\frac{24}{25}$ and $15061\frac{119}{121}$?

Answers: $2\frac{1}{6}$, $3\frac{2}{7}$, $18\frac{3}{5}$ and $122\frac{8}{11}$.

(8) Required the square roots of $32\frac{1}{9}$, $41\frac{1288}{1575}$, $\frac{4.41}{.64}$, $\frac{.00841}{1000}$ and $756.28\frac{16993}{20736}$.

Answers: $5\frac{2}{3}$, $6\frac{7}{15}$, 2.625, .0029 and $27.5\frac{1}{144}$.

(9) Determine the square roots of .9, 876.535, 728.6527 and 29.41275 to four places of decimals.

Answers: .9486, 29.6063, 26.9935 and 5.4233.

(10) What are the square roots of the recurring decimals, $.\dot{1}$, $.02\dot{7}$ and $.\dot{0}4938271\dot{6}$.

Answers: $.\dot{3}$, $.1\dot{6}$ and $.\dot{2}$.

EXTRACTION OF THE CUBE ROOT.

173. The Investigation of this operation is best conducted by general Symbols, and we shall merely put down here such observations and directions as are necessary and sufficient for performing it.

Digits:

1, 2, 3, 4, 5, 6, 7, 8, 9:

Cubes:

1, 8, 27, 64, 125, 216, 343, 512, 729:

and it is important that these last numbers and the corresponding roots should be committed to memory.

174. *Given the number of figures in a number, to find the number of figures in its cube root.*

Since, the cube root of 1 is 1:
the cube root of 1000 is 10:
the cube root of 1000000 is 100: &c.,

it follows that the cube root of a number between 1 and 1000 consists of one figure: that of a number between 1000 and 1000000 of two figures: that of one between 1000000 and 1000000000 of three figures, and so on; so that if a point be placed over every third figure, beginning at the *units'* place, the number of points thus placed will be that of the digits in the cube root: and it may manifestly be extended to Decimals.

Rule for the Extraction of the Cube Root.

Point the figures as above directed: then the *first* figure of the root is the number whose cube is equal to, or next less than, the *first* period on the left hand: and the remaining figures will be obtained by the following *uniform* process.

To the remainder, if any, bring down the *next* period, and for a *divisor* take 300 times the square of the part of the root already found: this gives the *next* figure of the root: perform the multiplication, to the product add the square of the *last* figure of the root when multiplied by the *rest* and by 30, and also the cube of the last, and subtract the sum: to the remainder annex* the next period, and proceed in the same way till the root or the requisite approximation to it is obtained.

The first and second *quotients* will frequently be taken too large; the remainder at any step must not *exceed* three times the square of the root obtained together with three times the root itself, and the number of figures in the root may nearly be *doubled* by ordinary division.

Ex. 1. Extract the cube root of 21952.

Here, after pointing the numbers, we have

$$
\begin{array}{rl}
 & 2\dot{1}95\dot{2}\ (28 = \text{cube root}: \\
2^3 = & 8 \\
2^2 \times 300 = 1200\,) & 13952 \text{ dividend}: \\
 & 9600 \\
8^2 \times 2 \times 30 = & 3840 \\
8^3 = & 512 \\
 & 13952 \text{ subtrahend}:
\end{array}
$$

and this is easily verified, for the cube of $28 = 21952$.

Ex. 2. Find the cube root of 12812.904.

$$
\begin{array}{rl}
 & 1\dot{2}81\dot{2}.90\dot{4}\ (23.4 = \text{cube root}: \\
2^3 = & 8 \\
2^2 \times 300 = 1200\,) & 4812 \text{ dividend}: \\
 & 3600 \\
3^2 \times 2 \times 30 = & 540 \\
3^3 = & 27 \\
 & 4167 \text{ subtrahend}: \\
23^2 \times 300 = 158700\,) & 645904 \text{ dividend}: \\
 & 634800 \\
4^2 \times 23 \times 30 = & 11040 \\
4^3 = & 64 \\
 & 645904 \text{ subtrahend}.
\end{array}
$$

Examples for Practice.

(1) Determine the cube roots of 1331, 15625, 46656 and 117649.

Answers: 11, 25, 36 and 49.

(2) Find the cube roots of 2197, 185193, 704969 and 912673.

Answers: 13, 57, 89 and 97.

(3) What are the cube roots of 33076161, 15069223, 105823817 and 873722816?

Answers: 321, 247, 473 and 956.

(4) Determine the cube roots of 17.576, 132.651, 493.039 and 64481.201.

Answers: 2.6, 5.1, 7.9 and 40.1.

(5) Required the cube roots of 18.609625, .007645373, .876467493 and .001030301.

Answers: 2.65, .197, .957 and .101.

(6) Extract the cube roots of $\frac{64}{343}$, $\frac{729}{140608}$, $49\frac{8}{27}$ and $7558\frac{197}{512}$.

Answers: $\frac{4}{7}$, $\frac{9}{52}$, $3\frac{2}{3}$ and $19\frac{5}{8}$.

(7) What are the approximate cube roots of .8, .27, and the exact cube roots of $.\dot{0}3\dot{7}$ and $1587.\dot{9}6\dot{2}$?

Answers: .9283 &c., .6463 &c., $.\dot{3}$ and $11.\dot{6}$.

EXTRACTION OF SOME OTHER ROOTS.

175. The directions already employed may by a little management be rendered available for the discovery of some other roots, as will be evinced in the following Examples.

Ex. 1. Required the fourth root of 1679616.

The fourth power of a quantity being equivalent to the square of its square, it is evident that the fourth root will be the same as the square root of its square root, and may be found by the two following operations performed according to the Rule laid down in Article (172):

```
     1̇ 6 7̇ 9 6̇ 1 6̇ (1296
     1
    -----
 22 ) 6 7                          1 2̇ 9 6̇ (36
      4 4                            9
     -------                        ------
249 ) 2 3 9 6                   66 ) 3 9 6
      2 2 4 1                        3 9 6
     ---------                      ------
2586 ) 1 5 5 1 6
       1 5 5 1 6
      ----------
```

and therefore the fourth root of 1679616 is 36.

Ex. 2. What is the sixth root of 308.915776?

Here, the *square* root is found to be 17.576: and the *cube* root of 17.576 is 2.6, which is evidently the *sixth* root of the quantity proposed.

176. What has been done in these two instances will serve to shew that all higher roots of quantities may be extracted by the rules already given, whenever the reciprocals of the indices representing them can be resolved into the factors 2 and 3 or these factors repeated: thus, the *eighth* root of 21035.8 = the square root of the fourth root of 21035.8 = the square root of the square root of the square root of 21035.8 = 3.47032 &c.; but such a process manifestly cannot be made use of in other cases.

SURDS OR IRRATIONAL QUANTITIES.

177. Def. When the quantity whose root is to be extracted is not a *complete* square, cube, &c., we have seen that there will be a remainder left however far we may continue the operation, and the root can therefore be found only *approximately*: that is, such a quantity has no *exact* root, and its representation is termed a *Surd* or *Irrational Quantity*.

For instance, the square root of 2 expressed by $\sqrt{2}$, is evidently not a *whole number*, because the square of no whole number whatever is 2: neither can it be a *vulgar fraction,* because the square of every vulgar fraction properly so called is itself a vulgar fraction; and it cannot be a *recurring decimal,* because all such quantities are equivalent to finite vulgar fractions: in other words, the square root of 2 may be found as nearly as we please, but not exactly; and it is termed an *incommensurable* quantity, because it admits of no exact measure which is any *finite* quantity whatever either integral or fractional.

178. The surds of most frequent occurrence are those designated by the sign $\sqrt[2]{\ }$ or $\sqrt{\ }$, or by the index $\frac{1}{2}$, and termed *Quadratic Surds:* and in general, when any *quantity* is represented in the form of a surd by means of a *fractional index,* it is always understood that the numerator of the index denotes the power to which the *number* is intended to be raised, and that the deno-

minator expresses the root afterwards to be extracted: thus, $27^{\frac{2}{3}}$ will represent the cube root of the square of 27, and is therefore equivalent to the cube root of 729 which is 9: that is, $27^{\frac{2}{3}}$, though expressed in the *form* of a surd, is in reality a rational quantity: and conversely.

179. Hence, the fundamental operations on surds must be performed upon their approximate values obtained as before: but these operations may frequently be shortened, as will appear in the following instances.

Since $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2}$, or $= 2\sqrt{2}$;

we have, in Addition, $\sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$:

in Subtraction, $\sqrt{8} - \sqrt{2} = 2\sqrt{2} - \sqrt{2} = \sqrt{2}$:

in Multiplication, $\sqrt{8} \times \sqrt{2} = 2\sqrt{2} \times \sqrt{2} = 4$:

in Division, $\sqrt{8} \div \sqrt{2} = 2\sqrt{2} \div \sqrt{2} = 2$:

where the extraction of only *one* root is sufficient for the operations of Addition and Subtraction, and the product and quotient are rational quantities.

The Involution and Evolution of such quantities may frequently be effected in the same way: thus, the square of $2\sqrt{5}$ = the product of the square of 2 and the square of $\sqrt{5} = 4 \times 5 = 20$ which is a rational number: and conversely.

Again, by multiplying each of the terms of the numerator and denominator by $\sqrt[3]{100}$, we have

$$\frac{\sqrt[3]{5.12} + \sqrt[3]{.03375}}{\sqrt[3]{80} - \sqrt[3]{.01}} = \frac{\sqrt[3]{512} + \sqrt[3]{3.375}}{\sqrt[3]{8000} - \sqrt[3]{1}}$$

$$= \frac{8 + 1.5}{20 - 1} = \frac{9.5}{19} = .5 = \frac{1}{2},\ \text{a rational quantity.}$$

180. It has been said that the *values* of surds may be found as nearly as we please: and this will clearly be done by continuing the extraction to the number of places of decimals in the root which we may find necessary for the purpose: thus, since $\sqrt{2} = 1.41421$ &c., we have,

$$\begin{aligned}\sqrt{2} &= 1.4, \text{ nearly}:\\ &= 1.41, \text{ more nearly}:\\ &= 1.414, \text{ still more nearly}:\\ &= 1.4142, \text{ still more nearly}:\\ &= \&\text{c.} \quad \ldots\ldots\ldots\ldots:\end{aligned}$$

and consequently its magnitude may be *compared* with that of any other numerical quantity either rational or irrational, although its *absolute* magnitude can never be exactly ascertained.

181. As quantities of this description have their origin in circumstances not purely *Arithmetical*, it is no objection to the definition of *Ratio* before given that that they scarcely seem to be included in it.

A ratio may however be incommensurable in *form*, when it is commensurable in *fact*, as is the case with the ratio $\sqrt{8} : \sqrt{2}$, whose magnitude is expressed by $2\sqrt{2} : \sqrt{2}$, or by $2 : 1$.

Again, because the ratio $\sqrt{3} : \sqrt{2}$ is the same with the ratio $\sqrt{3} \times \sqrt{2} : \sqrt{2} \times \sqrt{2}$, or $\sqrt{6} : 2$, the *magnitude* of this ratio may be found to every degree of nicety by continually increasing the number of decimal places in the extraction of the square root of 6.

The Arithmetic Mean between two numerical magnitudes being *half* their sum, will always be commensurable when they are so themselves; but the Geometric Mean, which is the *square root* of their product, will not necessarily be a terminating quantity under the same circumstances: thus, the Arithmetic Mean between 13 and 24 is 18.5, a rational quantity, whereas the Geometric Mean between them is $\sqrt{312} = 17.663$ &c., which is an incommensurable magnitude.

Example for Practice.

(1) Find the approximate values of $4 \times \left(\frac{5}{156}\right)^{\frac{1}{2}}$ and $\sqrt{3} \times (\sqrt{5} - 1)$, to four places of decimals.

Answers: .7161 and 2.1409.

(2) What are the sum and difference of $5\sqrt{2}$ and $7\sqrt{8}$, to four places of decimals?

Answers: 26.8698 &c. and 12.7278 &c.

(3) Find the value of the expression $16\sqrt{3} + 10\sqrt[3]{4} - 4\sqrt{12} - 3\sqrt[3]{108}$, to two places of decimals.

Answer: 15.43 &c.

(4) Determine the product and quotient of $5\sqrt{18}$ and $7\sqrt{63}$, to three places of decimals.

Answers: 1178.415 &c. and .381 &c.

(5) What is the square of $3\sqrt{7}$ and the cube of $\sqrt{2} \times \sqrt[3]{9}$?

Answers: 63 and $18\sqrt{2}$.

(6) Required the approximate values of the square roots of $\sqrt{11}$ and $14 - 6\sqrt{5}$, to two places of decimals.

Answers: 1.82 &c. and .76 &c.

(7) Which is the greater of $\sqrt{2} + \sqrt{7}$ and $\sqrt{3} + \sqrt{5}$? also, of $\sqrt{6} - \sqrt{5}$ and $\sqrt{8} - \sqrt{7}$?

Answers: $\sqrt{2} + \sqrt{7}$ and $\sqrt{6} - \sqrt{5}$?

(8) Reduce $\sqrt{20}$, $2\sqrt{45}$ and $3\sqrt{80}$, so that they may contain the same surd.

Answer: $2\sqrt{5}$, $6\sqrt{5}$, and $12\sqrt{5}$.

(9) Determine the exact value of $\sqrt{19 + 8\sqrt{3}} + \sqrt{19 - 8\sqrt{3}}$.

Answer: 8.

(10) Find the exact value of the compound surd $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.

Answer: 2.

These subjects are fully discussed by means of general symbols in the *fifth* Chapter of the Author's *Elements of Algebra*.

CHAPTER VIII.

THE NATURE AND PROPERTIES OF LOGARITHMS.

182. DEF. 1. *Logarithms* are a series of magnitudes increasing by a common *Difference*, corresponding to another series of magnitudes increasing by a common *Multiplier:* thus, if the former series be the natural numbers increasing by the common difference 1, as

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \&c.;$$

and the latter begin with 1 and increase by the common multiplier or factor 2, as

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,\ \&c.,$$

$$\text{or, } 2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ 2^6,\ 2^7,\ 2^8,\ 2^9,\ \&c.;$$

any term of the former series is *defined* to be the logarithm of the *corresponding* term of the latter: thus, we have

$$\begin{array}{ll} 0 = \log \text{ of } 2^0 \text{ or } 1; & 5 = \log \text{ of } 2^5 \text{ or } 32; \\ 1 = \log \text{ of } 2^1 \text{ or } 2; & 6 = \log \text{ of } 2^6 \text{ or } 64; \\ 2 = \log \text{ of } 2^2 \text{ or } 4; & 7 = \log \text{ of } 2^7 \text{ or } 128; \\ 3 = \log \text{ of } 2^3 \text{ or } 8; & 8 = \log \text{ of } 2^8 \text{ or } 256; \\ 4 = \log \text{ of } 2^4 \text{ or } 16; & 9 = \log \text{ of } 2^9 \text{ or } 512;\ \&c.; \end{array}$$

where the number 2, which has been *arbitrarily* assumed, is called the *Radix* or *Base* of the *System* of Logarithms: and it is evident that if the magnitude of any term in either of these series of quantities be assigned, that of the corresponding term in the other will be given.

Also, if an arithmetic mean between any two of the terms of the former series be found, it is manifest from the manner in which the two series are connected, that a geometric mean between the two corresponding terms of the second series must have the *same* relation to it, throughout the whole extent of both the series adopted.

A simpler idea of these numbers will perhaps be had by defining the logarithm of a magnitude to be the index of a *fixed* number which, when raised to the power denoted by that index, produces the magnitude, the fixed

number being assumed of any magnitude whatever, that of unity excepted because every power of 1 is 1.

183. Def. 2. If the number 10, which is the Base of the Common System of Notation, be adopted for the base of the logarithms as above defined, the terms of the series

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \&c.$$

will, by the last Article, be the logarithms of the corresponding terms of the series.

$$10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9, \&c.:$$

that is, in a system of logarithms whose base is 10,

$$\begin{aligned}
0 &= \log 10^0 \text{ or } 1; \\
1 &= \log 10^1 \text{ or } 10; \\
2 &= \log 10^2 \text{ or } 100; \\
3 &= \log 10^3 \text{ or } 1000; \\
4 &= \log 10^4 \text{ or } 10000; \\
5 &= \log 10^5 \text{ or } 100000; \\
6 &= \log 10^6 \text{ or } 1000000; \\
7 &= \log 10^7 \text{ or } 10000000; \\
8 &= \log 10^8 \text{ or } 100000000; \\
9 &= \log 10^9 \text{ or } 1000000000; \ \&c. = \&c.:
\end{aligned}$$

and it is further manifest from what has been said, that the arithmetic mean between any two terms of the first series will be the logarithm of the geometric mean between the two corresponding terms of the second.

The arithmetic mean of 0 and 1 is .5:
the geometric mean of 1 and 10 is 3.16227 &c.;
and therefore .5 = the logarithm of 3.16227 &c.

The arithmetic mean of .5 and 1 is .75:
the geometric mean of 3.16227 &c. and 10 is 5.62341 &c.;
whence .75 = the logarithm of 5.62341 &c.

The arithmetic mean of 1 and 2 is 1.5:
the geometric mean of 10 and 100 is 31.62277 &c.;
whence 1.5 = the logarithm of 31.62277 &c.:

and by continued repetitions of the process upon these and other numbers it follows that the logarithms of all

magnitudes whatever might be ascertained, though the labour requisite to do it would be immense.

It appears that 0 is the logarithm of 1 in any system whatever its base may be.

184. Def. 3. There is no difficulty in seeing that the logarithm of a magnitude between 1 and 10 will be a decimal fraction: that of a magnitude between 10 and 100 will be 1 with a decimal fraction annexed: that of one between 100 and 1000 will be 2 with a corresponding decimal fraction, and so on: for,

$$0.5 = \log 3.16227 \text{ \&c.}:$$
$$0.75 = \log 5.62341 \text{ \&c.}:$$
$$1.5 = \log 31.62277 \text{ \&c.}: \text{\&c.}:$$

and the integers 0, 1, 2, 3, &c., to the left of the decimal points in the logarithms of magnitudes are called the *Characteristics* of those logarithms: thus, 0 is the characteristic of the logarithms of all magnitudes between 1 and 10; 1 is the characteristic of the logarithms of all magnitudes between 10 and 100; 2 that of all magnitudes between 100 and 1000; &c.

185. Def. 4. If the logarithms of all magnitudes be calculated by processes analogous to the one above explained, (or indeed by any other methods which the present advanced state of mathematical science may suggest, but which were unknown to the more early writers upon the subject,) and the results be put into the form of a table, we shall have what is called a *Table of Logarithms;* and this may be used to facilitate the arithmetical operations of Multiplication, Division, Involution and Evolution, and to render these operations when applied to surds or other complicated magnitudes, exceedingly concise and easy. The advantages thus conferred upon the practical mathematician will be fully explained and exemplified in the following Articles.

186. *The Logarithm of the Product of two magnitudes is equal to the sum of the Logarithms of those magnitudes.*

Resuming the two series of magnitudes last used, we have

$$\begin{array}{lcccccccccc} \textit{logarithms,} & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, \&c.: \\ \textit{numbers,} & 1, & 10^1, & 10^2, & 10^3, & 10^4, & 10^5, & 10^6, & 10^7, & 10^8, & 10^9, \&c.: \end{array}$$

and in these we observe that

$$\begin{aligned} \log(1\times 10) &= \log 10 = 1 = 0+1 \\ &= \log 1 + \log 10: \\ \log(10\times 100) &= \log 1000 = 3 = 1+2 \\ &= \log 10 + \log 100: \\ \log(10\times 1000) &= \log 10000 = 4 = 1+3 \\ &= \log 10 + \log 1000: \\ \log(100\times 10000) &= \log 1000000 = 6 = 2+4 \\ &= \log 100 + \log 10000: \ \&c.: \end{aligned}$$

also, it is manifest from the formation of these numbers, that the same must be universally true, and that

$$\begin{aligned} \log 6 &= \log(2\times 3) = \log 2 + \log 3: \\ \log 15 &= \log(3\times 5) = \log 3 + \log 5: \\ \log 24 &= \log(4\times 6) = \log 4 + \log 6: \ \&c.: \end{aligned}$$

and this property may be rendered available to facilitate the multiplication of numbers whenever a table of logarithms, as explained in the last Article, is at hand.

Ex. Let it be required to find the product of the numbers 7 and 23, by means of a table of logarithms.

Here, referring to tables of this description, we find

$$\begin{aligned} \log 7 &= 0.8450980, \\ \log 23 &= 1.3617278, \end{aligned}$$

the characteristics which are *there* omitted being 0 and 1 respectively, for the reasons assigned in Article (184): whence, the logarithm of the required product will be

$$0.8450980 + 1.3617278 = 2.2068258;$$

and by looking again into the table, we find that this quantity without the characteristic, namely, .2068258 is the logarithm of 161, the characteristic itself merely shewing that the number is between 100 and 1000: that is, we have now the *logarithm* of the required product equal to the *logarithm* of 161, and consequently the product will be 161.

The operation above given may be more *conveniently* arranged as follows:

$$\begin{aligned}\log(7\times 23) &= \log 7 + \log 23\\ &= 0.8450980 + 1.3617278\\ &= 2.2068258\\ &= \log 161:\end{aligned}$$

and therefore $7 \times 23 = 161$, as we know to be the case.

Precisely in the same manner, whatever be the number of factors as 17, 26, 35, &c., we shall have

$$\begin{aligned}\log(17\times 26\times 35\times \&c.) &= \log 17 + \log(26\times 35\times \&c.)\\ &= \log 17 + \log 26 + \log 35 + \&c.,\end{aligned}$$

from which the product may be ascertained as in the preceding example.

187. *The Logarithm of the Quotient of two magnitudes is equal to the difference of the Logarithms of those magnitudes.*

Referring to the statement made at the head of the last Article, we see that

$$\begin{aligned}\log(10\div 1) &= \log 10 = 1 = 1 - 0\\ &= \log 10 - \log 1:\\ \log(1000\div 10) &= \log 100 = 2 = 3 - 1\\ &= \log 1000 - \log 10:\\ \log(1000000\div 100) &= \log 10000 = 4 = 6 - 2\\ &= \log 1000000 - \log 100: \ \&c.:\end{aligned}$$

and the general nature of these operations leads us to conclude similarly, that

$$\begin{aligned}\log 3 &= \log(6\div 2) = \log 6 - \log 2:\\ \log 9 &= \log(27\div 3) = \log 27 - \log 3:\\ \log 23 &= \log(161\div 7) = \log 161 - \log 7: \ \&c.\end{aligned}$$

This property will enable us to ascertain the quotient of two quantities, by the help of a logarithmic table.

Ex. What is the quotient arising from the division of 324 by 27?

Here,
$$\begin{aligned}\log(324\div 27) &= \log 324 - \log 27\\ &= 2.5105452 - 1.4313639\\ &= 1.0791813\\ &= \log 12:\end{aligned}$$

whence it follows from the equality of these logarithms, that

$$324 \div 27 = 12,$$

as is easily verified by ordinary division.

188. *The Logarithm of a Power of a magnitude is equal to the Logarithm of that magnitude multiplied by its index.*

For, we have seen in the preceding Articles, that

$$\log 10^2 = \log 100 \quad = 2 = 2 \times 1 = 2 \times \log 10:$$
$$\log 10^3 = \log 1000 \quad = 3 = 3 \times 1 = 3 \times \log 10:$$
$$\log 10^4 = \log 10000 = 4 = 4 \times 1 = 4 \times \log 10: \text{ \&c.}:$$

and similar conclusions will manifestly hold for the powers of any other magnitudes, as

$$\log 4^3 \quad = 3 \quad \times \log 4:$$
$$\log 9^7 \quad = 7 \quad \times \log 9:$$
$$\log 18^{10} = 10 \times \log 18: \text{ \&c.}$$

Ex. To find the seventh power of 2, we have

$$\begin{aligned} \log 2^7 &= 7 \times \log 2 \\ &= 7 \times 0.3010300 \\ &= 2.1072100 = \log 128: \end{aligned}$$

whence, *suppressing* the logarithms of both, we have

$$2^7 = 128,$$

which ordinary multiplication will shew to be true.

189. *The Logarithm of the Root of a magnitude is equal to the Logarithm of that magnitude divided by the whole number which denotes the root.*

For, as before, it is evident that

$$\log \sqrt{100} = \log 10 = 1 = 2 \div 2 = \tfrac{1}{2} \log 100:$$
$$\log \sqrt[3]{1000} = \log 10 = 1 = 3 \div 3 = \tfrac{1}{3} \log 1000:$$
$$\log \sqrt[5]{100000} = \log 10 = 1 = 5 \div 5 = \tfrac{1}{5} \log 100000: \text{ \&c.}:$$

and similarly, whatever the numbers may be, as

$$\log \sqrt{11} \quad = \tfrac{1}{2} \log 11:$$
$$\log \sqrt[4]{125} \quad = \tfrac{1}{4} \log 125:$$
$$\log \sqrt[9]{3421} = \tfrac{1}{9} \log 3421: \text{ \&c.}$$

Ex. To extract the seventh root of 128, we have

$$\begin{aligned}\log \sqrt[7]{128} &= \tfrac{1}{7}\log 128\\ &= \tfrac{1}{7}(2.1072100)\\ &= .3010300 = \log 2:\end{aligned}$$

whence is immediately obtained $\sqrt[7]{128} = 2$.

190. From the preceding Articles and examples given to illustrate them we perceive that by the assistance of a table of logarithms, the operation of *Multiplication* is reduced to that of *Addition:* the operation of *Division* to that of *Subtraction:* the operation of *Involution* to that of *Multiplication*, and the operation of *Evolution* to that of *Division:* and it cannot now be difficult to see of what immense importance such numbers must be in those departments of science wherein these operations are called into frequent practice, and more particularly in the use of surds or other complicated quantities, which it would require great labour to manage according to the rules previously laid down.

191. As far as the *theoretical* view of logarithms is concerned, it is manifestly of very little importance what magnitude be adopted as the base of the system: but in *practice*, the one here assumed may easily be shewn to possess great advantages over all others, both as to the computations of the numbers themselves, as well as to their practical use.

From the properties of these numbers taken notice of in Articles (186) and (187), it will appear that the logarithms of *all* magnitudes expressed by the *same* significant digits, whether they be *integral, decimal* or *mixed*, differ only in their characteristics, the quantity to the right of the decimal point called the *Mantissa* or *Over-weight,* remaining the same for them all.

For, by every multiplication or division of a quantity by the *Base*, the characteristic of its logarithm is increased or diminished by an *unit;* because we have

$$\begin{aligned}\log 1230 &= \log(123 \times 10)\\ &= \log 123 + \log 10 = \log 123 + 1:\\ \log 12300 &= \log(123 \times 100)\\ &= \log 123 + \log 100 = \log 123 + 2:\end{aligned}$$

$$\log 123000 = \log(123 \times 1000)$$
$$= \log 123 + \log 1000 = \log 123 + 3 : \ \&c.:$$
$$\log 12.3 = \log(123 \div 10)$$
$$= \log 123 - \log 10 = \log 123 - 1 :$$
$$\log 1.23 = \log(123 \div 100)$$
$$= \log 123 - \log 100 = \log 123 - 2 :$$
$$\log .123 = \log(123 \div 1000)$$
$$= \log 123 - \log 1000 = \log 123 - 3 : \ \&c.:$$

and since the characteristic of the logarithm of a figure in the place of units is 0, it is evident that the characteristic in any case will be *additive* or *subtractive* according as the number is *greater* or *less* than unity: and it is on this account that in the tables usually employed the characteristics are entirely omitted, being intended to be supplied by the calculator when wanted.

Thus, by means of a logarithmic table, we have

$$\log 123 = 2.0899052,$$

the characteristic 2 being here supplied from the considerations mentioned in (184): therefore from what is done above, we get

$$\log 1230 \quad = 1 + \log 123 = 3.0899052 :$$
$$\log 12300 \quad = 2 + \log 123 = 4.0899052 :$$
$$\log 123000 \quad = 3 + \log 123 = 5.0899052 :$$
$$\log 1230000 = 4 + \log 123 = 6.0899052 : \ \&c.:$$
$$\log 12.3 \quad = \log 123 - 1 = 1.0899052 :$$
$$\log 1.23 \quad = \log 123 - 2 = 0.0899052 :$$
$$\log .123 \quad = \log 123 - 3 = \bar{1}.0899052 :$$
$$\log .0123 = \log 123 - 4 = \bar{2}.0899052 : \ \&c.:$$

the small lines made over the 1 and 2 in the last two logarithms being intended to shew that the characteristic is there to be *subtracted*, instead of being *added* as in the rest, the mantissa remaining additive as before.

The construction of logarithmic tables will consequently be much facilitated by the adoption of the number 10 as their base, a single mantissa now belonging to all magnitudes expressed by the same significant digits, which evidently could not be the case were any other assumed in its stead: and the advantage arises entirely

from the circumstance of this number being the base of the system of notation in general use.

192. It would be foreign to the design of the present work to enter into the detail of the methods employed in the construction of a Table of Logarithms, and we shall merely notice, among some of the uses of such a table, how the logarithms of numbers may in certain cases be derived from one another, and what expedients may be resorted to in order to establish their correctness.

193. *To find the Logarithm of a Composite Number.*

Let the number be decomposed into its *prime* factors; then by Article (186), the logarithm of the number proposed is equal to the sum of the logarithms of its factors.

$$\text{Thus, since } 987 = 3 \times 329 = 3 \times 7 \times 47,$$

$$\text{we have } \log 987 = \log 3 + \log 7 + \log 47;$$

and if the latter be known, the first is found: also, these logarithms if calculated *independently* will verify one another.

194. *To find the Logarithm of a Fraction.*

Let the logarithm of the denominator be subtracted from the logarithm of the numerator, and the difference will be the logarithm of the proposed fraction, as appears from Article (187).

$$\text{Thus, } \log \frac{5}{7} = \log 5 - \log 7:$$

$$\text{and } \log 3\tfrac{4}{5} = \log \frac{19}{5} = \log 19 - \log 5:$$

and from these instances it follows that the logarithm of a *proper* fraction is subtractive whilst that of an *improper* fraction is additive.

In practice, the logarithm of a proper fraction is adjusted so as to have its mantissa *additive* and its characteristic *subtractive*, as in Article (191): thus,

$$\log \frac{5}{7} = \log 5 - \log 7 = \log 50 - 1 \quad \log 7 = \bar{1} + (\log 50 - \log 7).$$

195. *To find the Logarithm of a Power or a Surd.*

Multiply the logarithm of the quantity by its index whether *integral* or *fractional*, and the result will be the logarithm of the power or surd proposed, as is evident from Articles (188) and (189).

$$\text{Thus, } \log 7^2 = 2\log 7:$$

$$\text{and } \log\left(\frac{2}{9}\right)^{\frac{3}{5}} = \frac{3}{5}\log\frac{2}{9} = \frac{3}{5}(\log 2 - \log 9):$$

and the logarithm of a surd will therefore be greater or less than the logarithm of its root, according as the index is greater or less than 1.

196. *To find a fourth proportional to three given magnitudes.*

From the sum of the logarithms of the second and third magnitudes subtract that of the first, and the remainder will be the logarithm of the fourth proportional which may therefore be found by the tables.

Thus, let x be a fourth proportional to 3, 7 and 11: then since

$$3 : 7 :: 11 : x,$$

$$\text{we have } x = \frac{7 \times 11}{3};$$

$$\text{and therefore } \log x = \log 7 + \log 11 - \log 3.$$

197. *To find a mean proportional, or geometric mean between two given magnitudes.*

Divide the sum of the logarithms of the proposed quantities by 2, and the quotient will be the logarithm of their mean proportional.

Thus, if x be the mean proportional between 13 and 17, we have, by the definition of a mean proportional,

$$13 : x :: x : 17;$$

and therefore by Article (130), we obtain

$$x^2 = 13 \times 17 \text{ and } x = (13 \times 17)^{\frac{1}{2}};$$

$$\text{whence, } \log x = \frac{1}{2}(\log 13 + \log 17).$$

198. The preceding Articles shew us that in the formation of a set of Logarithmic Tables it will be

necessary to calculate the logarithms of the prime numbers only, and that those of their various multiples may then be found by addition.

When part of a table has thus been constructed, one portion of it may be used to verify another: thus when we have found the logarithms of 3, 5 and 6, we should have

$$1 = \log 10 = \log \frac{30}{3} = \log 30 - \log 3 = \log 5 + \log 6 - \log 3:$$

and by means such as these, a *check* may be applied at any stage of the process in order to ascertain the correctness of the previous computations.

199. For the reader's exercise we put down here the logarithms of the prime numbers less than 100 without their characteristics; and he will thus be enabled to construct for himself a table of the logarithms of all other numbers up to 100.

Nos.	Logarithms.	Nos.	Logarithms.
2	3010300	43	6334685
3	4771213	47	6720979
7	8450980	53	7242759
11	0413927	59	7708520
13	1139434	61	7853298
17	2304489	67	8260748
19	2787536	71	8512583
23	3617278	73	8633229
29	4623980	79	8976271
31	4913617	83	9190781
37	5682017	89	9493900
41	6127839	97	9867717

These logarithms are extracted from Mr. Babbage's Tables which every *practical* Student should have in his possession.

Examples for Practice.

(1) Required the logarithms of 5 and 168.

Answers: .6989700 and 2.2253093.

(2) Determine the logarithms of 1.04 and 3690.

Answers: .0170334 and 3.5670265.

(3) What are the logarithms of $1\frac{3}{4}$ and $\frac{8}{11}$?

Answers: .2430380 and $\bar{1}.8616973$.

(4) Express the logarithm of 225 by means of the logarithms of 2 and 3, and verify it.

Answer: $2 - 2\log 2 + 2\log 3$.

(5) Given the logarithms of 3 and 7, find the logarithm of 14700, and verify it.

Answer: $2 + \log 3 + 2\log 7$.

(6) Given the logarithms of 2 and 3, deduce the logarithm of .0072, and prove the converse.

Answer: $3\log 2 + 2\log 3 - 4$.

(7) Find the logarithm of 50000 in terms of the logarithms of 216 and .081.

Answer: $5.75 - \frac{1}{3}\log 216 + \frac{1}{4}\log .081$.

(8) Express the logarithms of 8 and 9 in terms of those of 6 and 15.

Answers: $1.5 + \frac{3}{2}\log 6 - \frac{3}{2}\log 15$ and $\log 6 + \log 15 - 1$.

(9) Find the logarithm of 83349 from the logarithms of 3 and .21.

Answer: $6 + 2\log 3 + 3\log .21$.

(10) Given the logarithms of 15 and 16, find those of 27 and $4\frac{1}{20}$.

Answers: $3\log 15 + \frac{3}{4}\log 16 - 3$ and $4\log 15 + \frac{3}{4}\log 16 - 5$.

(11) Find the logarithms of 15.625 and .00475.

Answers: $6\log 5 - 3$ and $2\log 5 + \log 19 - 5$.

(12) Required the logarithms of $\frac{9}{16}$ and $\frac{2}{375}$ in terms of the logarithms of 2, 3 and 5.

Answers: $2\log 5 + 2\log 3 - 2\log 2 - 2$,
and $4\log 2 - \log 3 - 3$.

(13) Determine the logarithms of $\sqrt[3]{\frac{24}{135}}$ and $\sqrt[4]{1.625}$, by means of those of 2, 3, 5 and 13.

Answers: $2\log 2 - \frac{2}{3}\log 3 + \frac{2}{3}\log 5 - 1$,

and $\frac{1}{4}\log 13 - \frac{3}{4}\log 2$.

(14) Express the logarithm of 7 in terms of the logarithms of 2 and $.\dot{7}1428\dot{5}$.

Answer: $1 - \log 2 - \log .\dot{7}1428\dot{5}$.

(15) Given the logarithm of $10424 = 4.0180353$: find the fifth root of $1\frac{3}{13}$.

Answer: 1.0424.

(16) Determine the value of the expression $\frac{2^6 \times 25^2}{4^3 \times 10^2}$ by means of logarithms.

Answer: 6.25.

(17) Find a fourth proportional to the quantities 1.3, .0104 and 2.375 by logarithms.

Answer: .019.

(18) Determine by logarithms a mean proportional between the magnitudes .004 and 72250.

Answer: 17.

(19) Given $.200686 = \log 1.58740 = 2\log 1.25992$: find the value of $\sqrt[3]{4} - \sqrt[3]{2}$.

Answer: .32748.

(20) Given $2.2309306 = \log 170.188$: it is required to find the value of $8 \times \sqrt[5]{7\sqrt{2} \times \sqrt[8]{3}}$.

Answer: 13.61504.

(21) Required the number of figures in the product of 324 and 126, by means of logarithms.

Answer: 5.

(22) Find the numbers of digits in the results of the involutions of 2^{10} and 3^{12}, by means of logarithms.

Answers: 4 and 6.

(23) Required by a table of logarithms, the index of 5 which shall give a result equal to 20.

$$\text{Answer: } \frac{1 + \log 2}{1 - \log 2}.$$

(24) Find the logarithm of 180 in a system whose base is 12, by means of a table of common logarithms.

$$\text{Answer: } \frac{1 + \log 2 + 2 \log 3}{2 \log 2 + \log 3}.$$

(25) Shew that the *Mantissa* of a logarithm depends upon the *figures* and not upon the *pointing off*: and that the *Characteristic* depends upon the *pointing off* and not upon the *figures*.

The invention of Logarithms is due to the celebrated JOHN NAPIER or NEPER, Baron of *Merchiston* in *Scotland*, who was born in the year 1550 and died in the 68th year of his age. The base of the *Napierian* System of Logarithms is the mixed magnitude 2.71828 &c.; but, for the great improvement in the subject hinted at in Article (191), we are indebted to Mr. HENRY BRIGGS, Professor of Geometry at *Oxford*, by whom a Table of Logarithms was published in the year 1624.

The reader who may be desirous of further information upon this portion of science is referred to Dr. HUTTON's Mathematical Tables which contain an account of the discoveries of the most celebrated writers connected with it: but he will not be able to appreciate their ingenuity and merits without a much more extensive knowledge of numerical calculations than can be acquired from this or any other treatise on *Arithmetic*.

CHAPTER IX.

THE APPLICATION OF ARITHMETIC TO GEOMETRY.

200. DEF. 1. In some of the preceding chapters the symbols and signs of *Pure Arithmetic* have been transferred from *abstract* magnitudes so as to represent *concrete* magnitudes and their *relations* to each other; and it is on the same principle that the objects of *Geometry* or *Geometrical Magnitudes*, as *Lines* or *Distances*, *Superficies* or *Areas*, and *Solid Contents* or *Volumes*, are valued and compared by means of the numbers representing their respective *Dimensions*.

A line having *length* only has *one* dimension: a superficies having *length* and *breadth* comprises *two* dimensions; and a solid has *three* dimensions, inasmuch as it is defined by three magnitudes, *length*, *breadth* and *depth* or *thickness*.

201. DEF. 2. A *Measure* in Geometry is a magnitude assumed as an *Unit* with which other magnitudes of the same kind may be compared: and though *one* magnitude neither contains *another* nor is contained in it an *exact* number of times, there may still be a *third* and *smaller* magnitude which is capable of *measuring* them both.

A measure has therefore the same relation to *quantity* as unit has to *number*; and all quantities and numbers are *said* to be *equal to* the aggregates or sums of their measures and units respectively.

When the magnitudes of lines are numerically expressed, the *Principles of Geometry* must furnish the means of valuing or comparing with each other, those of both superficies and solids of which lines naturally form the dimensions: and on this account we shall first establish the *Theory* of *Lineal Measure* and then deduce those of *Superficial* and *Solid Measure* from it.

THE THEORY OF LINEAL OR LONG MEASURE.

202. Def. An *Unit* of lineal or long measure is a straight line of a certain length, *arbitrarily* fixed upon; and by means of the *ratios* which other lines bear to it, their *numerical* magnitudes are ascertained.

Thus, if the straight line ab be the lineal unit, the numerical magnitude of the straight line AB will be

determined from the following proportion:

the magnitude of ab : the magnitude of AB :: *a* lineal unit : *the* lineal units in AB; that is,

the magnitude of AB will be

$$= \text{the magnitude of } ab \times \frac{\text{the lineal units in } AB}{\text{a lineal unit}}$$

$= \text{the magnitude of } ab \times \text{the } \textit{number} \text{ of lineal units in } AB$:

whence, representing the magnitude of ab by *unity* or 1, we shall have the numerical magnitude of AB represented by the *number* of lineal units contained in it; that is, if the lengths of two straight lines AB and CD be respectively 4 times and 6 times as great as the length of the lineal unit ab, the magnitudes of AB and CD will be 4 and 6 respectively which are expressed by the equalities,

$$AB = 4 \text{ and } CD = 6:$$

and the same method of proceeding will shew that if any straight line be a *multiple* of the lineal unit, the numerical representative of its magnitude will be a *whole* number.

If the line AB be not an *exact* multiple of the lineal unit ab but have a *common* measure with it, so that when they are *both* divided by it, the common measure is contained 7 times and 3 times in them respectively; then, we have

the magnitude of AB : the common measure :: 7 : 1;

and the common measure : the magnitude of ab :: 1 : 3;

that is, by Articles (130) and (131), we have

the magnitude of $AB = 7 \times$ the common measure;

and the common measure $= \frac{1}{3} \times$ the magnitude of ab;

from which equalities we find

$$\text{the magnitude of } AB = 7 \times \frac{1}{3} \times \text{the magnitude of } ab$$

$$= \frac{7}{3} \times \text{the magnitude of } ab$$

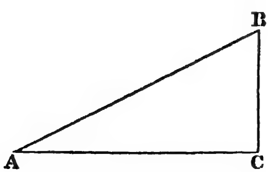
$$= \frac{7}{3},$$

the magnitude of ab being represented by 1, as before; and thus, whenever a line is not a *multiple* of the lineal unit but has a *common* measure with it, its numerical magnitude will be represented by a *fraction*.

If, however, the proposed line AB be neither a *multiple* of the lineal unit ab, nor have a *common* measure with it, as for instance, if $AB = \sqrt{2}$, then only an approximate arithmetical representation of its *value* can be had, where the approximation may easily be carried far enough to answer every practical purpose, as appears from Article (180).

It need scarcely be observed here, that if the lineal unit be an *inch*, a *foot*, a *yard*, &c., the corresponding magnitudes of the proposed lines will be expressed in inches, feet, yards, &c., and their parts respectively.

If the base AC and the perpendicular altitude BC of the triangle ABC right-angled at C, be



arithmetically represented by 4 and 3 denoting 4 inches and 3 inches respectively: then by EUCLID, I. 47,

$$AB^2 = AC^2 + BC^2$$

$$= 4^2 + 3^2 = 16 + 9 = 25:$$

whence, by extracting the square roots of both sides of the equality, we have

$$AB = 5 \text{ inches};$$

or AB will be arithmetically represented by 5.

If $AC = 3$ feet and $BC = 2$ feet, we have, by the same proposition,

$$AB^2 = AC^2 + BC^2 = 3^2 + 2^2 = 9 + 4 = 13:$$

and thence, by the extraction of the square root, we find

$$AB = \sqrt{13} = 3.605 \text{ \&c. feet},$$

which is only an approximation to the *true* value but may be continued to as much nicety as we please.

If we had $AC = BC = 1$ yard, then would

$$AB^2 = AC^2 + BC^2 = 1^2 + 1^2 = 2:$$

and therefore, by performing the same operation, we have

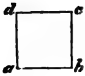
$$AB = \sqrt{2} = 1.4142135 \text{ \&c. yards}.$$

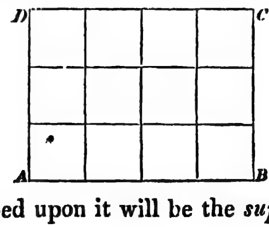
The last result proves the *hypotenuse* of a right-angled isosceles triangle, or the *diagonal* of a square, to be *incommensurable* with either of the *sides*.

Hence, it appears that a quadratic surd may be expressed accurately in Geometry though not so in Arithmetic; and it is also clear that any other *geometrical* proposition may be translated into the symbols of *Arithmetic* and any part determined, when the number of the *data* or of the *parts given*, is *sufficient* for the purpose.

THE THEORY OF SUPERFICIAL OR SQUARE MEASURE.

203. Def. An *Unit* of superficial or square measure is a square surface or area, whereof the length of each side is the lineal unit: thus, if ab be the *lineal* unit,



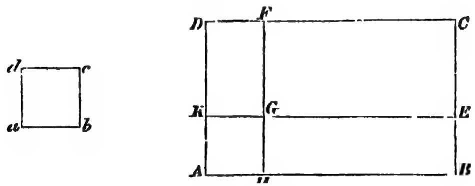


the square $abcd$ described upon it will be the *superficial*

or *square* unit, having the *two* dimensions ab and ad, which may be regarded as its *length* and *breadth:* and the magnitude of any proposed surface or area will be obtained by finding what *multiple, part* or *parts,* the surface or area is of *this* unit.

204. *The numerical representative of the Area of a rectangular parallelogram is equal to the product of those of two of its adjacent sides.*

Let $ABCD$ be a rectangular parallelogram whereof the adjacent sides AB and AD contain 7 and 5 lineal units respectively; take $AH = AK =$ the *lineal* unit, and draw KE and HF parallel to AB and AD intersecting in G, so that AG being equal to $abcd$,



will be the superficial unit: then, by EUCLID, VI. 1, we have

the area of the parallelogram $ABEK$: the area of the superficial unit $AHGK$ :: AB : AH :: 7 : 1;

whence, by Articles (130) and (131),

the area of the parallelogram $ABEK = 7 \times$ the area of the superficial unit $AHGK$;

again, by the same proposition we have

the area of the parallelogram $ABCD$: the area of the parallelogram $ABEK$ :: AD : AK :: 5 : 1;

or, the area of the parallelogram $ABCD = 5 \times$ the area of the parallelogram $ABEK$;

and therefore, from the preceding equality, we obtain

the area of the parallelogram $ABCD = 5 \times 7 \times$ the area of the superficial unit $AHGK$;

whence, if the area of the superficial unit be represented

by 1, the area of the parallelogram $ABCD$ will be represented by

$$5 \times 7 \text{ or } 35,$$

which is the product of two of its adjacent sides;

or, the area of the parallelogram $ABCD = AB \times AD$

$$= 7 \times 5 = 35 \text{ superficial units.}$$

Also, from the general principle of the demonstration, the same conclusion must hold good if the sides be represented by *fractions* or *irrational quantities*, inasmuch as the proposition of geometry here made use of has reference to *quantity* and not to *number* only.

If the two sides be equal to one another and to 12 inches or 1 foot, so that $ABCD$ becomes a square: then the area of the *square* $ABCD$

$$= AB \times AD = 12 \times 12 = 144;$$

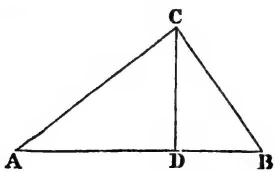
that is, if the side of a square contain 12 *lineal* inches, its area will comprise 144 *superficial* or *square* inches: or, in other words, 144 square inches are *equal* to 1 square foot.

Similarly, 9 square feet are equal to 1 square yard and $30\frac{1}{4}$ or 30.25 square yards, to 1 square pole.

Hence it follows from Euclid, I. 35 and 36, that the area of *any* parallelogram is expressed by the product of the numerical values of its base and altitude.

205. If the base and altitude of a triangle be represented by numerical magnitudes, its area will be numerically represented by half their product.

For, let the base AB be equal to 8 feet and the altitude CD to 3 feet:



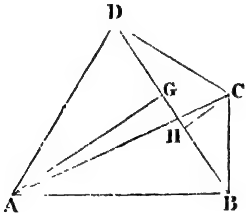
then, by EUCLID, I. 37 and 41, the area of the triangle

ABC is equal to *half* that of the parallelogram whose base is AB and whose altitude is CD: whence,

$$\text{the area of the triangle } ABC = \tfrac{1}{2} \text{ of } AB \times CD$$
$$= \tfrac{1}{2} \text{ of } (8 \times 3) = \tfrac{1}{2} \text{ of } 24 = 12;$$

that is to say, if the base and altitude of a triangle be equivalent to 8 and 3 *lineal* units respectively, then will its area be represented by 12 *superficial* units of the same name; and it is of no consequence whether the dimensions be integral, fractional or irrational, as appears from Article (204).

206. If we take the four-sided figure $ABCD$ called a *trapezium*, and



find the numerical value of the *diagonal* BD and of each of the *perpendiculars* AG and CH let fall upon it from the angles A and C, the area of the figure being the sum of the areas of the two triangles ABD and BCD, may be ascertained.

Thus, if by measuring we have found that $BD = 5$, $AG = 4$ and $CH = 1\frac{1}{2}$ lineal units; we shall have

$$\text{the area of } ABCD = \text{the area of } ABD + \text{the area of } BCD$$
$$= \tfrac{1}{2}BD \times AG + \tfrac{1}{2}BD \times CH$$
$$= \tfrac{1}{2}(5 \times 4) + \tfrac{1}{2}(5 \times 1.5)$$
$$= \tfrac{20}{2} + \tfrac{7.5}{2} = 10 + 3.75$$
$$= 13.75 = 13\tfrac{3}{4} \text{ superficial units};$$

and the same result must evidently have been obtained if perpendiculars had been let fall upon the *other* diago-

nal AC from the angles B and D, because the area of the figure cannot have *two* different magnitudes.

Similarly, the area of any rectilineal figure may be found by adding together the areas of the triangles which compose it.

207. Conversely, if the area of a parallelogram or of a triangle and either its base or altitude be given, the other of these magnitudes will be obtained by *division.*

Also, if the superficial units comprised in the area of a square whose *side* is AB, be 1521; then,

$$AB^2 = 1521:$$

from which, by the extraction of the square root, we have

$$AB = 39:$$

that is, if the area of a square surface be 1521 *superficial* units, each of its sides will be 39 *lineal* units.

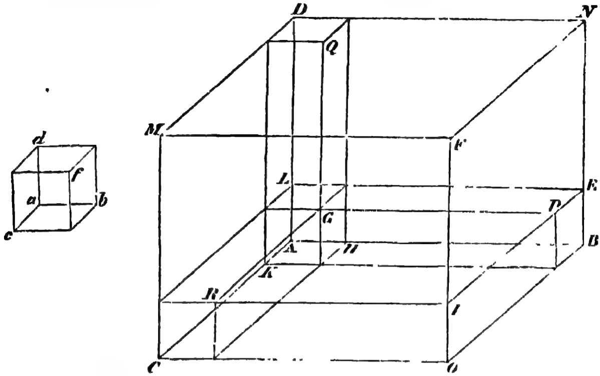
Again, an acre, being a parallelogram 40 poles in length and 4 poles in breadth containing 4840 square yards, will be equal to a square whose side $= \sqrt{4840} = 69.57$ &c. $= 69\frac{4}{7}$ yards, nearly.

THE THEORY OF SOLID OR CUBIC MEASURE.

208. Def. An *Unit* of solid or cubic measure is a cube or rectangular parallelepiped whose length, breadth and thickness are each equal to the lineal unit; as the solid af represented hereafter, wherein $ab = ac = ad =$ the lineal unit, denotes the *solid* or *cubic* unit: and the solid content or volume of any body of *three* dimensions will be ascertained by finding what multiple, part or parts, it is of *this* unit, the lineal dimensions or the length, breadth and thickness being supposed first to be numerically exhibited.

209. *The numerical representative of the Solid Content or Volume of a rectangular parallelepiped is equal to the product of the magnitudes representing its length, breadth and thickness.*

Let $ABFM$ represent a rectangular parallelepiped whereof the length $AB = 5$, the breadth $AC = 4$ and the thickness $AD = 3$ lineal units, the *denominations* of the dimensions being the same in each:



take $AH = AK = AL =$ the lineal unit and complete the construction as in the diagram; then AG will be a cube equal to the solid unit af; and by Euclid, xi. 25, we have

$$\text{the parallelepiped } AF : \text{the parallelepiped } AI$$
$$:: \sqcap CD : \sqcap CL :: AD : AL :: 3 : 1;$$

and the parallelepiped $AF = 3 \times$ the parallelepiped AI; also, the parallelepiped AI : the parallelepiped AP

$$:: \sqcup BI : \sqcap BP :: AC : AK :: 4 : 1;$$

and the parallelepiped $AI = 4 \times$ the parallelepiped AP; again, the parallelepiped AP : the parallelepiped AG

$$:: \sqcap BL : \sqcap HL :: AB : AH :: 5 : 1;$$

and the parallelepiped $AP = 5 \times$ the parallelepiped AG; whence, we have the parallelepiped AF

$$= 3 \times \text{the parallelepiped } AI$$
$$= 3 \times 4 \times \text{the parallelepiped } AP$$
$$= 3 \times 4 \times 5 \times \text{the parallelepiped } AG;$$

but the parallelepiped AG being equal to the solid unit is represented by 1; consequently, the numerical magni-

tude of the rectangular parallelepiped whose three contiguous edges are 3, 4 and 5 *lineal* units will be represented by

$$3 \times 4 \times 5 = 60 \textit{ solid} \text{ units}:$$

that is, the content of the parallelepiped $ABFM$

$$= AB \times AC \times AD = 5 \times 4 \times 3 = 60.$$

If the three edges AB, AC, AD be equal to one another and their magnitude be 3 *lineal* feet or 1 yard, the parallelepiped becomes a cube whose magnitude $= 3 \times 3 \times 3 = 27$ *solid* feet: that is, 27 solid or cubic feet are *equal* to 1 solid or cubic yard.

Similarly, 1728 cubic inches are equal to 1 cubic foot.

Also, it follows, from Euclid, XI. 31, that the content of *any* parallelepiped is equal to the product of the area of its base and its altitude.

210. Hence may be found the length of an edge of the cube which is of equal solid content with any proposed parallelepiped.

Thus, if a parallelepiped be 7 inches in length, $3\frac{1}{2}$ inches in breadth and $1\frac{3}{4}$ inches in depth, its solid content will be

$$7 \times 3\tfrac{1}{2} \times 1\tfrac{3}{4} = 42\tfrac{7}{8} \text{ inches}:$$

whence, the edge $= \sqrt[3]{42\frac{7}{8}} = \sqrt[3]{42.875} = 3.5 = 3\frac{1}{2}$ inches.

In the same manner the altitude of a parallelepiped may be found by dividing the solid content by the area of the base; and *vice versâ*.

211. It will not here be necessary to pursue these subjects further; and we shall now only give directions for ascertaining the measures of such magnitudes as most frequently present themselves to our notice, without attempting their investigations which belong to higher departments of Mathematics.

THE PRACTICE OF LINEAL MEASURE.

(1) *Right-angled Triangle.* The square root of the sum of the squares of the *Sides* forming the right

angle is equal to the *Hypotenuse:* and the square root of the difference of the squares of the hypotenuse and *either* side is equal to the *other* side.

(2) *Circle.* The circumference is equal to the product of twice the radius by 3.14159, nearly: and the radius is equal to the quotient of the circumference by 6.28318, nearly.

(3) Hence, the *Homologous Lines* in similar triangles and in all circles are *proportional.*

Ex. 1. If the base of a triangle be 1, and the perpendicular be 1, the hypotenuse $= \sqrt{1^2 + 1^2} = \sqrt{2}$.

If the base be $\sqrt{2}$ and the perpendicular be 1, the hypotenuse $= \sqrt{2+1} = \sqrt{3}$.

If the base be $\sqrt{3}$ and the perpendicular be 1, the hypotenuse $= \sqrt{3+1} = \sqrt{4} = 2$.

If the base be 2 and the perpendicular be 1, the hypotenuse $= \sqrt{4+1} = \sqrt{5}$, and so on: and in all these, only approximate *arithmetical* values of the surds can be found by evolution; also, it is worth noticing how all the *primitive* surds successively originate from these *geometrical* considerations, as has been hinted before at the end of Article (202).

Ex. 2. The wheels of a carriage are $2\frac{1}{2}$ yards asunder and the inner wheel describes the circumference of a circle whose radius is 20 yards: find the difference of the paths of the two wheels.

The circumference of the inner circle $= 3.14159 \times 40$: the circumference of the outer circle $= 3.14159 \times 45$: whence, their difference will evidently $= 3.14159 \times 5 = 15.70795$ yards $= 15\frac{7}{10}$ yards, nearly.

Examples for Practice.

(1) Required the hypotenuse of a right-angled triangle whose sides are 24 and 32 feet.

Answer: 40 feet.

(2) Find the diameter of a rectangle whose sides are .2 of an inch and .99 of an inch.

Answer: 1.01 inches.

(3) Find the base of the right-angled triangle, whose other sides are 4 and $\sqrt{48}$.

Answer: $4\sqrt{2}$.

(4) If a ladder 103.44 feet long be placed so as to reach a window 40 feet high on one side of a street and a window 60 feet high on the other side: what is the breadth of the street?

Answer: 180 feet, nearly.

(5) Of two ships from the same port one has sailed 50 leagues due east and the other 84 leagues due north: what is their distance from each other?

Answer: $97\frac{3}{4}$ leagues, nearly.

(6) A square field has a diagonal path across it, measuring 125 yards: find the length of its side.

Answers: 88.3883 yards, nearly.

(7) Find the circumference of a circle whose radius is 6.3662 yards.

Answer: 40 yards, nearly.

(8) If the diameter of the Earth be 7912 miles, find the length of a *French Metre* which is one ten-millionth part of a fourth part of its circumference.

Answer: 39.37231584 inches, nearly.

(9) Shew that $\frac{22}{7}$, $\frac{333}{106}$ and $\frac{355}{113}$ are approximations to the known numerical value of the circumference of a circle whose diameter is 1, and point out which is the nearest.

THE PRACTICE OF SUPERFICIAL MEASURE.

(1) *Parallelogram.* The area is equal to the product of the base and the altitude.

(2) *Triangle.* The area is equal to half the product of the base and the altitude.

(3) *Triangle.* From half the sum of the three sides, subtract each side separately: multiply together the half-sum and the three remainders, and the square root of the product will be equal to the area.

(4) *Trapezium.* The area is equal to half the product of either diagonal and the sum of the perpendiculars let fall upon it from the opposite angles.

(5) *Circle.* The area is equal to the square of the radius multiplied by 3.14159, nearly.

(6) *Sector.* The area is equal to half the product of the radius and the subtending arc.

(7) *Ellipse.* The area is equal to the product of the semi-axes multiplied by 3.14159, nearly.

(8) *Cylinder.* The convex surface is equal to the product of the circumference of the base and the altitude.

(9) *Right Cone.* The convex surface is equal to the product of the circumference of the base and half the slant height.

(10) *Sphere.* The convex surface is equal to four times the area of a circle of the same radius.

(11) *Spherical Segment.* The convex surface is equal to the product of the circumference of a circle of the same radius and the altitude.

(12) Hence, the *Areas* of similar figures are as the *squares* of their homologous lineal dimensions.

Ex. 1. Find the area of a triangle whose sides are 18, 24 and 30 poles.

Here, we have according to the directions above, half the sum of the three sides $= \frac{1}{2}(18 + 24 + 30) = 36$:

$$\text{also, } \left.\begin{aligned} 36 - 18 &= 18 \\ 36 - 24 &= 12 \\ 36 - 30 &= 6 \end{aligned}\right\} \text{ are the three remainders:}$$

whence, the area $= \sqrt{36 \times 18 \times 12 \times 6} = \sqrt{46656} = 216$ square poles.

Ex. 2. If the radius of a circle be 2 feet, find the side of the square whose area shall be equal to it.

The area of the circle $= 4 \times 3.14159 = 12.56636$ square feet, nearly: whence, by Article (207), the side of the required square $= \sqrt{12.56636} = 3.545$ feet, nearly.

Examples for Practice.

(1) If the sides of a triangle be 16.6, 18.32 and 28.6: find its area.

Answer: 143, nearly.

(2) If the diagonal of a trapezium be 498 yards and the perpendiculars let fall upon it from the opposite angles be 10.8 and 18.8 yards: what is its area?

Answer: 7370.4 yards.

(3) Each side of a hexagon is 24 feet and the perpendicular upon each side from a certain point within it is $12\sqrt{3}$ feet: find its area.

Answer: $864\sqrt{3}$ feet.

(4) Find the sides of the squares whose areas are 4970.25 square inches and $885\frac{1}{16}$ square feet.

Answers: 70.5 inches and $29\frac{3}{4}$ feet.

(5) How much must be cut off from a rectangular surface $2\frac{1}{4}$ feet broad to make a square yard?

Answer: 4 feet.

(6) If two acres of land be laid out in the form of a circle, what is its radius?

Answer: $55\frac{1}{2}$ yards, nearly.

(7) Find the radius of a circle whose area is equal to that of a square whose side is 5.317 yards.

Answer: 3 yards, nearly.

(8) The semiaxes of an ellipse are 25 and 49: find the radius of a circle of equal area.

Answer: 35.

(9) The base of a triangle is 14.1 yards and its area is 64.86 yards: find its height.

Answer: 9.2 yards.

(10) The side of an equilateral triangle is 6: find its area.

Answer: 15.588, nearly.

(11) The two equal sides of an isosceles triangle are 12 feet and the base is 8 feet; required its area.

Answer: 45.2548 feet, nearly.

(12) Compare the area of a circle with the area of the square inscribed in it.

Answer: 3.14159 : 2, nearly.

(13) What is the relation between the area of a square and that of the circle inscribed in it?

Answer: 4 : 3.14159, nearly.

(14) Required the area of the sector of a circle whose arc and radius are each 2.57 inches.

Answer: 3.30245 inches.

(15) The radii of two concentric circles are 10 and 12 yards: find the space included between them.

Answer: 138.22996 yards, nearly.

(16) Prove that the convex surface of a right cone is found by multiplying together the circumference of its base and half its slant height.

THE PRACTICE OF SOLID MEASURE.

(1) *Parallelepiped.* The content is equal to the area of the base multiplied by the altitude.

(2) *Prism and Cylinder.* The content is equal to the area of the base multiplied by the altitude.

(3) *Pyramid and Cone.* The content is equal to the area of the base multiplied by one third of the altitude.

(4) *Sphere or Globe.* The content is equal to the cube of the radius multiplied by 4.18879, nearly.

(5) Hence, the *Contents* of similar solid bodies are as the *cubes* of their homologous lineal dimensions.

Ex. 1. Required the depth of a parallelepiped $29\frac{2}{3}$ long and $44\frac{1}{2}$ broad, so that its content shall be equal to that of a cube whose edge is 89.

Here, the area of the base of the parallelepiped

$$= 29\tfrac{2}{3} \times 44\tfrac{1}{2} = \frac{89 \times 89}{3 \times 2} = \frac{89^2}{6}:$$

whence, the depth of the parallelepiped

$$=89^3 \div \frac{89^2}{6} = \frac{89^3 \times 6}{89^2} = 89 \times 6 = 534.$$

Ex. 2. The content of a cylinder is equal to the sum of the contents of a cone and hemisphere having the same base and altitude.

Taking 1 to represent the radius of the hemisphere, we shall have immediately from the directions contained in the last page:

the content of the hemisphere = 2.09439, nearly:

the content of the cone = 1.04719, nearly:

the content of the cylinder = 3.14159, nearly:

whence, we find the sum of the *two former*

= 3.14158 nearly, which is the *last* very nearly:

and this would have been an *exact equality* were it not for the circumstance of *each* of the contents being only an *approximation* to its true value.

Examples for Practice.

(1) Each edge of the base of a square prism is 34 inches and its height is $12\frac{5}{12}$ feet: find its content.

Answer: 99 feet, 1172 in.

(2) What weight of water will a cistern contain, the length being 4 ft., the breadth 2 ft. 6 in. and the depth 3 ft. 3 in. when a cubic foot of water weighs 1000 ounces?

Answer: 32500 ounces.

(3) A rectangular cistern whose length is $9\frac{3}{4}$ feet and breadth is 6 feet contains $294\frac{1}{4}$ cubic feet: find its depth.

Answer: $5\frac{7}{234}$ feet.

(4) What length of a cylindrical stone roller 18 inches in diameter, must be taken to make 14.137155 solid feet?

Answer: 2 feet.

(5) The sides of the base of a triangular pyramid are 3, 4 and 5 feet and its altitude is 6 feet: find its solid content.

Answer: 12 feet.

(6) The solid content of a sphere is two thirds of that of its circumscribed cylinder.

(7) A right cone, hemisphere and cylinder of the same base and altitude are as the numbers 1, 2, 3.

(8) A sphere is equal to a cone whose height is equal to the radius and whose base is equal to the area of four great circles of the sphere.

THE COMPUTATIONS OF ARTIFICERS.

212. DEF. *Artificers* take the dimensions of their work in *yards, feet, inches, parts, &c.:* and it is usual to reduce the yards to feet so that the different denominations are *all* connected by the same number 12, or decrease in a *twelvefold* ratio, from the place of feet towards the right hand. For the sake of uniformity, the denominations after feet are termed *primes, seconds, thirds, &c.*, distinguished respectively by accents $'$, $''$, $'''$, &c., placed a little to the right, contiguous to the figures to which they belong: thus, 20 feet, 8 inches, 5 parts, &c., is written $20^f . 8' . 5''$. &c.

The operation employed to compute superficial and solid contents is that of Multiplication, conducted by means of a mixed *Decimal* and *Duodecimal* scale of Notation; the figures of the feet being expressed and multiplied in the ordinary way, whilst in the other places the number 12 is always made use of instead of 10. The denomination on the left hand of the multiplier is used first, those of the multiplicand being taken as in other cases; then the next in order, and so on: and for the reason that we put the first figure of a *partial* product one place to the *left* of that of the preceding one when we begin with the least denomination of the multiplier, the terms of the product here must each be put one place to the *right* of those of the preceding, in order to possess their proper relative values: and the addition is effected by beginning with the lowest denomination, as in compound quantities.

From the circumstance above mentioned, the process is sometimes called *Cross Multiplication;* and it is also frequently termed *Duodecimal* Multiplication or *Duodecimals:* but these latter names are evidently misapplied,

because the *different* digits of the various denominations are not connected with each other by the number 12 though the *denominations* themselves are. The practical applications of the rule will be best taught by examples.

Ex. 1. Find the area of a rectangular parallelogram whose adjacent sides are 5 ft. 3 in. and 4 ft. 9 in. (See fig. p. 198.)

Here, $AB = 5^f . 3'$ and $AD = 4^f . 9'$:

whence, the area $= 5^f . 3' \times 4^f . 9'$: and the multiplication is effected in the following *form*:

$$\begin{array}{lll} 5^f . \ 3' & & = \text{length:} \\ 4 \ . \ 9 & & = \text{breadth:} \\ \hline 21 \ . \ 0 & & = \text{product by } 4^f: \\ \ \ 3 \ . \ 11 \ . \ 3 & & = \text{product by } 9' = \tfrac{9}{12}f: \\ \hline 24^f . \ 11 \ . \ 3'' & & = \text{area:} \end{array}$$

and precisely as in a product in the *common scale* of notation, the denomination of 11′ is a *twelfth* part of a square foot, which is called a *superficial prime*: that of 3″ is a *twelfth* part of a superficial prime, termed a *superficial second*: and so on, if there were more terms: so that the area expressed in square *feet* is

$$24 + \frac{11}{12} + \frac{3}{144} = 24 + \frac{135}{144} = 24 \text{ sq. feet } 135 \text{ sq. inches.}$$

This result may easily be verified by either *Vulgar Fractions* or *Decimals*: thus,

by Vulgar Fractions, the area $= 5\tfrac{1}{4} \times 4\tfrac{3}{4} = \dfrac{21}{4} \times \dfrac{19}{4} = \dfrac{399}{16}$

$$= 24\tfrac{15}{16} = 24\tfrac{135}{144} \text{ sq. feet, as above:}$$

by Decimals, the area $= 5.25 \times 4.75 = 24.9375$

$$= 24 \text{ sq. feet } 135 \text{ sq. inches, as before.}$$

Ex. 2. Required the area of a square whose side is $7^f . 8' : 9''$.

The operation here requisite will be the following:

$$\begin{array}{l}
7^f\,.\,8'\,.\,9'' \\
7\,.\,8\,.\,9 \\
\hline
54\,.\,1\,.\,3 \\
5\,.\,1\,.\,10\,.\,0 \\
\quad 5\,.\,9\,.\,6\,.\,9 \\
\hline
59^f\,.\,8'\,.\,10''\,.\,6'''\,.\,9'''' \\
\hline
\end{array}$$

or, the area is 59 feet, 8 primes, 10 seconds, 6 thirds and 9 fourths, all superficial measure: and expressed in square feet, it will be

$$59 + \frac{8}{12} + \frac{10}{144} + \frac{6}{1728} + \frac{9}{20736} = 59\,\frac{15345}{20736} = 59\,\frac{1705}{2304}$$

square feet: and the square inches, square parts, &c., might be found by the ordinary reductions.

Ex. 3. Find the content of a rectangular parallelepiped whose lineal dimensions are 5ft. 6in., 4ft. 5in. and 3ft. 4in. (See fig. p. 202.)

Here, $AB = 5$ ft. 6 in., $AC = 4$ ft. 5 in., $AD = 3$ ft. 4 in.: and we have the following operation:

$$\begin{array}{l}
5^f\,.\,6' \\
\\
22\,.\,0 \\
2\,.\,3\,.\,6 \\
\hline
24\,.\,3\,.\,6 \\
3\,.\,4 \\
\hline
72\,.\,10\,.\,6 \\
8\,.\,1\,.\,2\,.\,0 \\
\hline
80^f\,.\,11'\,.\,8''\,.\,0''' \\
\hline
\end{array}$$

or, the content is 80 solid feet, 11 solid primes and 8 solid seconds, which expressed in solid feet is

$$80 + \frac{11}{12} + \frac{8}{144} = 80\,\frac{140}{144} = 80\,\frac{1680}{1728}$$

= 80 solid feet 1680 solid inches.

Ex. 4. Required the capacity of a cube the length of whose edge is 2 feet 9 inches.

The capacity $= 2^f.\,9' \times 2^f.\,9' \times 2^f.\,9' = 7^f.\,6'.\,9'' \times 2^f.\,9'$

$= 20^f.\,9'.\,6''.\,9''' = 20 + \frac{9}{12} + \frac{6}{144} + \frac{9}{1728} = 20\frac{1377}{1728}$ cubic feet $=$ 20 cubic feet 1377 cubic inches; and it may easily be verified by vulgar fractions or decimals.

213. The method of computation just explained is exceedingly simple and well adapted to the use of *Workmen.* The reverse operations of division and evolution are not often required by them, and though they might be conducted on the same plan, it will be much easier to express the quantities *fractionally* or *decimally,* and then to proceed according to the ordinary methods.

The *Prices* of Artificers' work, being at so much per foot, yard, &c., may be calculated by the *Rule of Proportion* or by the *Rules of Practice.*

Examples for Practice.

(1) Multiply 14ft. 6in. by 12ft. 7in.
Answer: $182^f.\,5'.\,6'' =$ 182 sq. ft. 66 sq. in.

(2) Multiply 25ft. 7in. by 7ft. 10in.
Answer: $200^f.\,4'.\,10'' =$ 200 sq. ft. 58 sq. in.

(3) Multiply 16ft. 5in. by 12ft. 11in.
Answer: $212^f.\,7'' =$ 212 sq. ft. 7 sq. in.

(4) Multiply $11^f.\,11'$ by $2^f.\,3'.\,4''$.
Answer: $27^f.\,1'.\,8''.\,8'''$.

(5) Multiply $9^f.\,4'.\,7''$ by $5^f.\,6'.\,4''$.
Answer: $51^f.\,10'.\,4''.\,0'''.\,4''''$.

(6) Multiply $17^f.\,3'.\,4''$ by $19^f.\,5'.\,11''$.
Answer: $336^f.\,9'.\,6''.\,8'''.\,8''''$.

(7) Multiply $10'.\,3''.\,4'''$ by $5'.\,0''.\,6'''$.
Answer: $4'.\,3''.\,9'''.\,9^{iv}.\,8^{v}$.

(8) Multiply $13^f.\,2'.\,6''$ by $1'.\,9''.\,10'''$.
Answer: $2^f.\,4'''.\,7''''$.

(9) Find the square yards, &c., in a plane rectangular surface 15ft. 5in. long and 9ft. 10in. broad.

Answer: 16yds. 7ft. 86in.

(10) How many squares of 100 feet are contained in a floor 19yds. 3in. long and 9yds. 1ft. 6in. broad?

Answer: 16sq. 31ft. 90in.

(11) Find the cost of a slab 5ft. 7in. long and 3ft. 8in. broad, at 3*s*. per square foot.

Answer: £3. 1*s*. 5*d*.

(12) Required the price of the carpet of a room 18ft. 6in. long and 14ft. 5in. broad, at 5*s*. per square yard.

Answer: £7. 6*s*. $5\frac{1}{2}$*d*.

(13) What is the value of a piece of building ground 34ft. 9in. by 26ft. 4in., at 1*s*. per square foot?

Answer: £45. 15*s*. 1*d*.

(14) How many square feet of paper will cover the walls of a room which is 20ft. 10in. long, 16ft. broad and 10ft. 9in. high?

Answer: 791 sq. ft. 132 sq. in.

(15) Find the whole surface of a room 22ft. 5in. long, 18ft. 4in. broad and 11ft. 8in. high.

Answer: 1772 sq. ft. 112 sq. in.

(16) How many square rods of $272\frac{1}{4}$ feet, are there in a rectangular piece of bricklayer's work whose dimensions are 15ft. and 68ft. 9″?

Answer: $3\frac{3}{4}$ sq. rods.

(17) Find the area of a triangle whose three sides are 2ft. 3in., 3ft. and 3ft. 9in.

Answer: 3 sq. ft. 54 sq. in.

(18) Determine the volume of a cube whose edge is 3ft. 10in. in length.

Answer: 56 solid ft. 568 solid in.

(19) Find the capacity of a rectangular cistern whose dimensions are 4ft. 6in., 5ft. 7in. and 6ft. 8in.

Answer: $167\frac{1}{2}$ cubic feet.

(20) The area of the base of a parallelepiped is 124ft. 28in. and its altitude is 8ft. 4in.: find its content in feet and inches.

Answer: 1034ft. 1648in.

(21) A parallelepiped contains 94ft. 235in. and the area of its base is 24ft. 5in.: find its altitude.

Answer: 3ft. 11in.

(22) Find the numbers of feet and inches in the side of a square whose area is 1122ft. 36in.

Answer: 33ft. 6in.

(23) Find the edge of a cube which contains 15 solid feet and 1080 solid inches.

Answer: 2ft. 6in.

(24) The area of one side of a cube is $12^f. 3'$; find its capacity.

Answer: 42 cubic ft. 1512 cubic in.

(25) Divide $1532^f. 9'. 9''$. superficial measure by $81^f. 9'$ lineal measure.

Answer: 18ft. 9in.

(26) How much in length that is 1ft. 2in. broad and $1\frac{1}{2}$in. thick, will make a solid foot?

Answer: 6ft. $10\frac{2}{7}$in.

(27) A gardener has a piece of matting 73yds. 1f. 8in. long and 3ft. 9in. broad, to cover a wall 94ft. long and 10ft. high: how much of the wall will be left uncovered?

Answer: $112\frac{1}{2}$ft.

(28) On laying down a plot of ground with sods 2ft. 6in. long and 9in. wide, it is found that it requires 75 sods to form one strip extending its whole length, and that a man can lay down $1\frac{1}{4}$ strips each day: find the surface covered in 8 days.

Answer: $1406\frac{1}{4}$ft.

(29) The roller used for a bowling-green being 6ft. 6in. in circumference and 2ft. 3in. in width, is observed to make 12 revolutions from one extremity of

the green to the other: find the area rolled, when the roller has passed 10 times the length of it.

Answer: 195yds.

(30) A reservoir is 24ft. 8in. long and 12ft. 9in. wide: how many cubic feet of water must be drawn off to make the surface sink 1 foot?

Answer: $314\frac{1}{2}$ft.

(31) If a cubical block of stone contain 14706ft. 216in.: find the length of all its edges, and the area of all its faces.

Answers: 294ft. and $3601\frac{1}{2}$ft.

THE COMPUTATIONS OF GAGERS.

214. Def. The dimensions made use of by *Gagers* are taken in *inches* and parts of an inch expressed *decimally:* and from them, the contents of cisterns, malt-bins, &c., are computed by such rules as have been already laid down, and they will therefore be expressed in cubic inches and their decimal parts.

215. *Liquids* are always estimated by the imperial *Gallon* which is equal to 277.274 cubic inches: and therefore, when the content of a vessel has been ascertained in cubic inches, the number of gallons it contains will be found by dividing it by 277.274.

Ex. What number of gallons are contained in a cistern whose length is 40 inches, breadth 24 inches and depth 16 inches?

Here, the content $= 40 \times 24 \times 16 = 15360$ cubic inches:

$$\text{whence, the number of gallons} = \frac{15360}{277.274}$$

$= 55.3964$ &c. gals. $=$ 55gals. 1qt. 1pt., nearly.

216. *Malt, Corn,* &c., are always estimated by the imperial *Bushel* consisting of 2218.192 cubic inches; and the number of bushels will be obtained by dividing the content, ascertained as before, by 2218 192.

Ex. If a circular room 5 feet in radius, be filled with malt to the depth of 6 inches: find the number of bushels it contains.

Here, the content $= 3.14159 \times 60^2 \times 6$

$= 67858.344$ cubic inches:

$$\text{and the number of bushels} = \frac{67858.344}{2218.192}$$

$= 30.5917$ &c. bush. $= 30$ bush. $2\frac{1}{3}$ pks., nearly.

Whenever the depth is an *inch*, the *content* of any upright vessel or cistern is expressed by the *area* of its surface: and in this sense the term *surface* is used in gaging.

Examples for Practice.

(1) How many gallons are contained in a cubic foot?

Answer: 6.232 gallons, nearly.

(2) The length of a cistern is 169 inches and the breadth is 125 inches: how many gallons does it contain, the liquor being 4 inches deep?

Answer: 304.752 gallons, nearly.

(3) If the interior edge of a cubical vessel be 1ft. 3in., how many imperial gallons will it hold?

Answer: 12.172 gallons, nearly.

(4) What is the content of a cylindrical vessel, the radius of whose base is 20 inches and height 54 inches?

Answer: 244.733 gallons, nearly.

(5) What number of bushels are contained in the space of a cubic yard?

Answer: 21.033 bushels, nearly.

(6) How many bushels of malt are there on a floor $5\frac{1}{2}$ feet by 4 feet, when its depth is 14 inches?

Answer: 20 bushels, nearly.

(7) If the exterior dimensions of a closed rectangular bin be 3ft. 5in., 2ft. 4in. and 1ft. 3in., find the quantity of malt it will contain, if the thickness of the material of the bin be 1in.

Answer: 6.816 bushels, nearly.

(8) The diameter of the base of a standard bushel measure is 18.789 inches: find its height.

Answer: 8 inches, nearly.

The practical calculations of *Excisemen* are greatly facilitated by means of an instrument called a *Sliding Rule* and by *Tables* containing the proper multipliers

and divisors for *Squares, Circles*, &c., which may be seen in any work treating *expressly* upon the subject.

THE COMPUTATIONS OF LAND-SURVEYORS.

218. DEF. The dimensions of land or of any surface of considerable extent, are taken by means of *Gunter's Chain* which is 4 *poles* or 22 *yards* in length and is divided into 100 equal parts called *Links*.

219. Since an acre is equal to a parallelogram, 40 poles or 10 chains in length and 4 poles or 1 chain in breadth, it will contain $1000 \times 100 = 100000$ square links; and therefore, if the lineal dimensions be expressed in links and the superficial contents be found, these results when divided by 100000 or with *five* figures *cut off* towards the right hand, will be the numbers of acres.

A lineal pole being $5\frac{1}{2}$ yards or 25 links, the magnitude of a square pole will be

$5\frac{1}{2} \times 5\frac{1}{2} = 30\frac{1}{4}$ sq. yards; or, $25 \times 25 = 625$ sq. links:

so that a pole $= 625$ sq. links or $= 30.25$ sq. yds.:

a rood $= 25000$ sq. links or $= 1210$ sq. yds.:

an acre $= 100000$ sq. links or $= 4840$ sq. yds.

Hence also, the magnitude of a square mile

$= 1760 \times 1760 = 3097600$ sq. yds. $= 640$ acres.

Ex. The length of a rectangular field being 25 chains 8 links and its breadth 14 chains 75 links: what number of acres does it contain?

Here, 25 chains 8 links = 2 5 0 8 links,

14 ... 75 ... = 1 4 7 5 ...,

```
             12540
            17556
           10032
           2508
acres  3 6.9 9 3 0 0
                   4
roods   3.9 7 2 0 0
                  40
poles  3 8.8 8 0 0 0
```

and therefore the field comprises 36 ac. 3 ro. $38\frac{22}{25}$ po.

Hence also, if the length of a field containing 36 ac. 3 ro. $38\frac{22}{25}$ po. be 25 chains 8 links, its breadth will be found by reversing the operation upon these magnitudes when expressed in links: thus,

$$\text{the breadth} = \frac{3699300}{2508} = 1475 \text{ links} = 14 \text{ chains } 75 \text{ links.}$$

Examples for Practice.

(1) Find the area of a square field whose side is $10\frac{1}{2}$ chains.

Answer: 11 ac. 4 po.

(2) The base of a triangular field is 16 chains 3 poles and its perpendicular is 6 chains 2 poles: what number of acres does it contain?

Answer: 5 ac. 1 ro. 31 po.

(3) The sides of a triangular field are 380, 420 and 765 yards: how many acres are contained in it?

Answer: 9 ac. 38 po., nearly.

(4) The diagonal of a trapezium is $5\frac{1}{4}$ chains and the perpendiculars upon it from the opposite angles are 3 and $2\frac{1}{2}$ chains: find its area.

Answer: 1 ac. 1 ro. 31 po.

(5) A field in the form of an ellipse has its greatest and least diameters equal to 7 and 5 chains: find how many acres there are in each of the parts into which they divide it.

Answer: 2ro. 30po., nearly.

(6) One acre of land is to be cut from a rectangular field whose breadth is 2 chains 50 links, by a line parallel to it: find the length of the plot.

Answer: 4 chains.

(7) What is the length of the side of a square field comprising 2 acres 4 poles?

Answer: $4\frac{1}{2}$ chains.

(8) The base of a triangular field is 11.313708 chains: find the length of the line parallel to the base, which divides it into two equal parts.

Answer: 8 chains.

(9) A square field contains 40 acres: determine the number of chains in its circuit. Find the area of a field of the same circuit, whose breadth is $\frac{3}{7}$ths of its length.

Answers: 80ch. and 33ac. 2ro. 16po.

(10) A foot-path of uniform breadth around a circular field contains as much as $\frac{9}{16}$ths of the field: compare the breadth of the path with the radius of the field.

Answer: 1 : 4.

IMPERIAL WEIGHTS AND MEASURES.

220. The Weights and Measures made use of in this Kingdom having from length of time become subject to certain irregularities, in addition to the want of uniformity which generally prevailed in them, and the *Standard* Weights and Measures being at best but vaguely defined, the subject was at length laid before a Board of Commissioners: and, in accordance with a report furnished by them, *An Act of Parliament* which came into operation on the first of January 1826 was passed, establishing an uniform System of *Imperial Weights* and *Measures,* the leading features of which we shall now endeavour to place before the reader.

221. Def. The *Standard* of Lineal Measure is a rod or beam whose length is called a *yard* which is equivalent to 3 *feet* or 36 *inches:* and the Standard *Square* and *Cubic* Measures will therefore depend entirely upon it, as has been seen in the preceding pages.

At present we have no means of ascertaining why this particular length was originally fixed upon; but as it is most essential that it should always remain the same, it will be found convenient to refer it to something else which we have no reason to suppose ever undergoes any change.

Now the length of a *Pendulum* vibrating *seconds* or performing 86400 *oscillations* in the interval between

the Sun's leaving the meridian of a place and returning to it again, is always the same at a *fixed* place and under the *same* circumstances: and if this length be divided into 391392 equal parts, the yard is *defined* to be 360000 of these parts: also, conversely, since a yard is equal to 36 inches, it follows that the length of the seconds' pendulum expressed in *inches* is 39.1392.

The pendulum referred to in this country is one vibrating seconds at *Greenwich* or in *London* at the *level of the sea* in a *non-resisting medium:* and should the *standard yard* at any time be lost, it would be easy to have recourse to experiment for its recovery.

The standard yard being the general *Unit* of lineal measure, it follows that all lengths less than a yard will be expressed by *fractions:* and it is on this account that a lineal *Inch* or *ten thousand* of the aforesaid portions of the pendulum is conveniently adopted as the *unit* of lineal measure when applied to small magnitudes.

Hence also, the standard superficial and solid measures will be accurately ascertained and kept correct.

222. DEF. The *Imperial Gallon* is the standard unit of the measure of capacity and is *defined* to be 277.274 cubic inches, the lineal inch being that above determined. The gallon and its multiples and parts are used to measure *liquids,* as Water, Spirits, &c., and *dry goods,* as Malt, Corn, &c., and the system is termed *The Imperial Liquid* and *Dry Measure.*

The *Imperial Bushel* consisting of *eight* gallons will consequently be 2218.192 cubic inches, and the *form* of the measure is defined by *Act of Parliament:* it is to be an upright cylinder whose internal diameter is 18.789 inches and depth 8 inches: but this can be of no importance when *heaped* measures are abolished.

The *Act of Parliament* directs that the *Heaped Imperial Bushel* shall be an upright cylinder whose diameter is not less than *twice* its depth, and that the height of the conical heap shall be at least *three fourths* of the depth, the boundary of its base being the outside of the measure: also, in heaped measure it is ordered that

3 *Bushels* make 1 *Sack,*
12 *Sacks*............1 *Chaldron:*

and the bushel is to be equal to 2815.4887 cubic inches.

223. Def. The *Imperial Pound Avoirdupois*, which is the standard unit by means of which all heavy goods of large masses are weighed is *defined* to be the weight of one *tenth* part of an imperial gallon or of 27.7274 cubic inches of distilled water, ascertained at a time when the barometer stands at 30° and the height of *Fahrenheit's* thermometer is 62°.

If the weight of a cubic inch of distilled water be divided into 505 equal parts and each of such parts be defined to be a *Half Grain*, it follows that 27.7274 cubic inches contain very nearly 7000 such grains; and it is declared by *Act of Parliament* that 7000 *Grains* exactly shall be considered as the *Pound* Avoirdupois, so that according to the *Table*, 1oz. = $437\frac{1}{2}$grs. and 1dr. = $27\frac{11}{32}$grs.

A cubic foot of *pure water*, at a temperature 60°, weighs very nearly 1000 ounces; and for *practical purposes*, this fact should be remembered.

This weight receives its name from *Avoirs* the ancient name for *goods* or *chattels* and *Poids* signifying *weight*, in the language of the country at the time of the *Normans*.

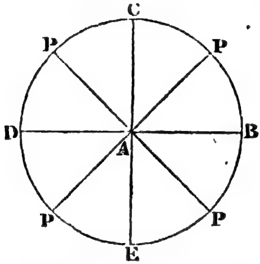
224. Def. The *Imperial Pound Troy* is *defined* to be 5760 grains determined as in the last Article: and this weight, we are told in the "Report of the Commissioners of Weights and Measures," is retained, because all the *Coinage* has been uniformly regulated by it and all *Medical Prescriptions* or *Formulæ* now are and always have been estimated by Troy Weight, under a peculiar subdivision which the *College of Physicians* are most anxious to preserve.

It is also stated that there are reasons to believe that the word *Troy* has been derived from the Monkish name of *Troy-novant*, given to *London*, founded on the Legend of *Brute*, which is mentioned by some of the old English writers. The story avers that *Brute* a lineal descendant of *Æneas* about the year of the world 2855 founded the city of *London*, then called *Trinovantum*, which was afterwards corrupted into *Trenovant* or *Troy-novant*.

225. Hence, 1lb. avoirdupois : 1lb. troy :: 7000 : 5760, or as 175 : 144; thus, 1lb. avoirdupois = $\frac{175}{144}$lbs.

troy = 14oz. 11dwts. 16grs. troy, and 1lb. troy = $\frac{144}{175}$lbs. avoirdupois = 13oz. $2\frac{114}{175}$drs.: and the *sub-divisions* of each pound are easily compared.

226. Def. The *Unit* of *Angular Measure* is called a *Degree* and is written or marked 1°: thus,



if the right angle *BAC* be supposed to be divided into 90 equal parts, each of them is called a degree and is considered to be the angular unit: and if the angle *BAP* be equal to any assigned number of such parts, as for instance 45, this angle will be 45° on the same scale that the right angle *BAC* is 90°: and thus the magnitudes of two or more angles may be compared.

If with the angular point *A* as a centre and any assumed radius *AB*, a circle be described and the diameters *BAD*, *CAE* be drawn at right angles to each other, dividing the circumference into four equal parts, *BC*, *CD*, *DE*, *EB* called *Quadrants;* then by Euclid vi. 33, the arcs *BC*, *BP* have the same ratio to each other as the angles *BAC*, *BAP* which they respectively subtend; and it is on this principle that a quadrantal arc is said to contain 90 degrees and therefore the whole circumference to be equal to 360 degrees; and although an angle and an arc being *heterogeneous* magnitudes cannot have the *same* unit of measure, it is clear that the division of the right angle into 90 equal portions of *angular measure* will *correspond* to the division of the quadrantal arc into 90 equal portions of *lineal measure*.

Thus, an angle and its subtending arc being *proportional* to each other will be connected by an *invariable* factor when the radius is given, so that *either* of them being known, the *other* may be ascertained.

That is, if the radius of the circle be the *lineal unit,* the circumference = 6.28318, nearly, by Article (211):

and therefore the quadrantal arc = 1.57079, nearly:

whence, we have $\angle BAC : \angle BAP :: \text{arc } BC : \text{arc } BP$,

$$\text{or, } \angle BAP = \angle BAC \times \frac{\text{arc } BP}{\text{arc } BC}$$

$$= \frac{90^\circ}{1.57079} \times \text{arc } BP, \text{ nearly,}$$

$$= 57^\circ.2957795 \times \text{arc } BP, \text{ nearly:}$$

and therefore the magnitude of the $\angle BAP$ is found by multiplying $57^\circ.2957795$ by the number of *lineal* units contained in the corresponding arc:

also, $57^\circ.2957795 = 57^\circ . 17' . 44'' . 48'''$, nearly,

must consequently be regarded as the *unit* of angular measure in all cases where an angle is to be determined by means of the *number* of lineal units in its arc, or, which is the same thing, by the *number* of radii to which its arc is considered equivalent.

THE CALENDAR.

227. Def. The interval of time between two passages of the Sun across the meridian of any place when taken at its *mean magnitude,* is termed a *Day* or a *Mean Solar Day,* which is supposed to be divided into 24 equal portions called *Mean Solar Hours.* It appears from the observations and calculations of Astronomers that the time between the Sun's leaving a fixed point in his path called the *Ecliptic* and returning to it again, consists of 365.242264 such days or of 365 days, 5 hours, 48 minutes, $51\frac{3}{5}$ seconds, very nearly, which is therefore termed a *Solar Year:* and thus the solar year, as here defined, does not consist of an exact number of solar days but is always expressed in days by a fractional quantity.

For the purposes of civil life it would be exceedingly inconvenient that one year should commence at one time of the day and another at a different time: and this circumstance gave rise to the invention of the *Civil Year* which the next three Articles will explain.

228. When the Science of Astronomy was much less perfect than it is at present, the length of the solar year was much less accurately known; and accordingly we find that in the time of *Julius Cæsar* it was supposed to consist of 365 days 6 hours, or of $365\frac{1}{4}$ days, *exactly*. On this supposition, it is evident that if out of *four* years in succession, any *three* consisted of 365 days each and the remaining one of 366 days, the Sun would have returned at the end of these *four* years to the place in the ecliptic which it occupied at their commencement.

The scheme was called the *Julian Calendar;* and if the hypothesis had been correct it would have been attended with much convenience: the additional day was made by *repeating* the *Sixth* of the *Calends of March* in the Roman Calendar, which corresponds with the 24th of *February* in ours: also, the *year* in which it was inserted was termed *Bissextile* and the additional *day* was called *Intercalary.*

This regulation applied to the years of the *Christian Æra,* was so managed that whenever the number of years was divisible by 4, the corresponding year consisted of 366 days and was called *Leap-year,* the month of *February* having 29 days in that year and in each of the remaining three years 28 days, without interfering at all with their order.

Hence also, the remainder after the division of any other number of years by 4, was the number of years since a leap year occurred up to that year: thus, in the year 1849 this remainder is 1; and accordingly it is 1 year since the last leap year happened and it is 3 years before the next will occur, according to this scheme.

229. Since the true solar year is 365.242264 days, and not 365.25 days, it is evident that the reckoning of time according to the Julian Calendar would place the end of the year *after* the time when the Sun had returned to the point of the ecliptic occupied by it at the beginning of the year and consequently in *advance* of the course of the *seasons:*

but, the error in one year is

$$365.25 - 365.242264 = .007736 \text{ of a day}:$$

whence, by finding how often this is contained in 1 day,

$$\frac{1}{.007736} = 129.2657\text{, nearly,}$$

will be the number of years in which the error amounts to 1 day: also, by the rule of proportion, we have

129.2657 yrs. : 400 yrs. :: 1 day : 3.0944 days, nearly :

and therefore 3.0944 days or 3 days, 2 hours, 16 minutes, is very nearly the error which would accumulate in 400 years.

Now, according to the Julian Calendar 400 years would comprise 100 leap years; and since we find that this reckoning falls nearly 3 days *after* the true time, if there were only 97 leap years in 400 years, the Julian year would very nearly agree with the true solar year; and it is accordingly ordained that whenever the *numbers* expressing the *Centuries*, as 16, 17, 18, 19, &c., denoting 1600, 1700, 1800, 1900, &c., are *not* divisible by 4, the corresponding year shall *not* be a leap year, although according to the Julian Computation it would: as, 1600 would be a leap year, but 1700, 1800, 1900, would not.

The Calendar thus corrected, though not absolutely accurate, is well adapted to every *practical* purpose, as the error in 5000 *years* will not amount to much more than *twenty-eight* hours. This correction was first promulgated in Europe by *Pope Gregory* in the year 1582, and the calendar has since been called the *Gregorian Calendar*, but it was not introduced into *Protestant Countries* till a much later period. In *England* it was adopted on the *second* day of September 1752 when the error amounted to 11 days: and it is called the *New Style* to distinguish it from the *Julian Calendar* which is now termed the *Old Style.*

Had the old style continued, the error would now have been 12 days, because 1800 would according to it have been a leap year which in the new it was not: and thus, we have in Almanacks, *Old Christmas-day, Old Midsummer-day*, &c., taking place *twelve* days after the times in which they are fixed by our present system.

Though all the calculations of *modern times* are conducted according to the new style, a knowledge of

the difference of the two styles is not without its use both in the *Perusal* of old Documents, and in the Astronomical *Verification* of Historical Facts which could not be performed without it.

230. The civil year thus fixed and determined is then subdivided into twelve Calendar Months, as described in the *Table*. The word *Month* however is used in different senses: sometimes to denote a *twelfth* part of the year or $30\frac{5}{12}$ days; sometimes as equivalent to 4 weeks or 28 days: and accordingly a year is equivalent to 13 months and 1 day, or, to 52 weeks and 1 day, with the addition of another day when it happens to be leap year.

FRENCH IMPERIAL MEASURES, &c.

231. In consequence of the irregularity in the measures and multiples of all the units just mentioned, it is evident that the calculation of measures and weights will be much less simple, particularly to *Foreigners*, than if they were connected by some common *factor;* and it was with the view of obviating this inconvenience, that a *New System* of measures and weights has been adopted in *France*.

232. In this system, the length of the Terrestrial Arc from the Equator to the Pole in the Meridian of *Paris* is taken as the *General Standard;* and the following Synopsis of *French Measures* exhibits them as compared with the standards of this country.

Lineal Measure.

Millimetre . . =	.03937	English Lineal Inches:
Centimetre . . =	.39370	. . . .
Decimetre . . =	3.9370	. . . .
Metre . . . . =	39.370	. . . .
Decametre . . =	393.70	. . . .
Hectometre. . =	3937.0	. . . .
Kilometre . . =	39370	. . . .
Myriometre. . =	393700	. . . .

Superficial Measure.

Are	= 119.5991	English Superficial Yards:
Decare . . .	= 1195.991	. . . .
Hectare . . .	= 11959.91	. . . .

Solid Measure.

Decistere . .	= 3.5315	English Solid Feet:
Stere	= 35.315	. . . .
Decastere . .	= 353.15	. . . .

Measure of Capacity.

Millilitre . .	= .06102	English Cubic Inches:
Centilitre . .	= .61024	. . . .
Decilitre . .	= 6.1024	. . . .
Litre . . .	= 61.024	. . . .
Decalitre . .	= 610.24	. . . .
Hectolitre . .	= 6102.4	. . . .
Kilolitre . .	= 61024	. . . .
Myriolitre . .	= 610238	. . . .

Here, the *Metre* or *one ten millionth part* of the Terrestrial Arc is the Element of lineal measure; the *Are* or Square *Decametre*, that of superficial measure; the *Stere* or Solid *Metre*, that of solid measure, and the *Litre* or Cubic *Decimetre*, that of the measure of capacity.

233. The weights in this system expressed in English grains, are

Milligramme . .	= .015432 . .	English Grains:
Centigramme . .	= .154326 .	. . . .
Decigramme . .	= 1.54326 .	. . . .
Gramme . . .	= 15.4326 .	. . . .
Decagramme . .	= 154.326 .	. . . .
Hectogramme .	= 1543.26 .	. . . .
Kilogramme . .	= 15432.6 .	. . . .
Myriogramme .	= 154326 .	. . . .

Here the *Gramme* is the Element, being the weight of a cubic *Centimetre* of distilled water.

234. The angular Measures in the same system expressed in English Degrees are as follow:

Second . . . =	.00009	English degrees:
Minute . . . =	.009	. . . .
Grade . . . =	.9	. . . .

Here, 100 Grades are consequently equivalent to 90 Degrees in the English scale: and in the inferior denominations, the *Centesimal* scale is uniformly used by the *French* where the *English* proceed according to the *Sexagesimal* scale.

235. The unit of value in France is a silver coin called a *Franc* consisting of $\frac{9}{10}$ths of pure silver and $\frac{1}{10}$th of alloy: and its subdivisions are as follow:

10 Centimes = 1 Decime:
10 Decimes = 1 Franc.

The value of an English pound sterling is equivalent to that of 25.2 francs, very nearly: and thus, the value of a franc expressed in English money is

$$\frac{240}{25.2} = 9.5238d., \text{ or } 9\tfrac{1}{2}d., \text{ very nearly.}$$

236. Wherever this system is used, the Theory of Decimals as laid down in the fifth Chapter will be sufficient for performing all the fundamental operations of Arithmetic, entirely superseding what has been done in the second Chapter of this work.

The Student who may be desirous of prosecuting his inquiries in this important subject is referred to the Articles, *Weights* and *Measures* in BARLOW's *Mathematical and Philosophical Dictionary*, and to the last edition of Dr. KELLY's *Universal Cambist.*

PROBLEMS.

237. We will conclude the Application of Arithmetic to Geometry with the consideration of a few Problems, in the solutions of which the *Principles* explained in this chapter are generally taken for granted.

(1) If two persons A and B start at the same time from two towns C and D distant 300 miles from each

other: when and where will they meet, if they travel at the respective rates of 7 and 8 miles an hour?

Since the rate or velocity of A is 7 miles an hour, and the rate or velocity of B is 8 miles an hour, therefore

$$7 + 8 = 15 \text{ miles}$$

is the distance by which they approach each other in 1 hour, or their *relative* velocity: hence we have

$$15 \text{ mi.} : 300 \text{ mi} :: 1 \text{ hr.} : 20 \text{ hrs.};$$

or, 20 hrs. is the time in which they approach 300 miles towards each other and therefore meet: whence the required time is expressed by the whole distance divided by the sum of their velocities or rates per hour.

Also, $7 \times 20 = 140$ miles is the distance travelled by A,
and $8 \times 20 = 160$ miles is the distance travelled by B:

and they meet at 140 and 160 miles from C and D respectively.

Hence also, the distance between them after any assigned interval may be found.

When two motions in a straight line are in opposite directions, the velocity of approach or the *relative* velocity is equal to the *sum* of the *absolute* velocities.

(2) A, travelling at the rate of 12 miles an hour starts 15 miles behind B who travels only 10 miles an hour: find when A will overtake B and the distance then travelled by each.

Here, the gain of A upon B is $12 - 10 = 2$ miles in 1 hour which is their relative velocity: whence,

$$2 \text{ mi.} : 15 \text{ mi.} :: 1 \text{ hr.} : 7\tfrac{1}{2} \text{ hrs.},$$

and $7\frac{1}{2}$ hrs. is the time in which A gains 15 miles upon B and therefore overtakes him: so that

$$A \text{ has travelled } 12 \times 7\tfrac{1}{2} = 90 \text{ miles},$$

$$B \text{ has travelled } 10 \times 7\tfrac{1}{2} = 75 \text{ miles}:$$

and the difference of these distances is 15 miles which is accordant with the enunciation of the problem.

In cases like this, the velocity of approach or *relative* velocity is the *difference* of the *absolute* velocities, and the time is found by dividing the distance by it.

The reasoning employed in these two instances is evidently applicable to any *uniform* motions whether in straight lines or curves, provided the distance be measured along the paths described.

(3) Two couriers pass through a place at an interval of 4 hours travelling at the rates of $11\frac{1}{2}$ and $17\frac{1}{2}$ miles an hour: how long and how far must the first travel before he is overtaken by the second?

The relative velocity $= 17\frac{1}{2} - 11\frac{1}{2} = 6$ miles:

also, $11\frac{1}{2} \times 4 = 46$ miles = the distance between them, when the second passes through the given place: whence, as before, we have $\frac{46}{6} = 7\frac{2}{3}$ *hrs.*, the time when the second after leaving the given place overtakes the first: and therefore the first has travelled in $11\frac{2}{3}$ hours, the distance $11\frac{1}{2} \times 11\frac{2}{3} = 134\frac{1}{6}$ *miles*: and the second in $7\frac{2}{3}$ hours, the distance $17\frac{1}{2} \times 7\frac{2}{3} = 134\frac{1}{6}$ *miles*.

(4) If 252 men in 5 days of 11 hours each, can dig a trench 210 yards long, 3 wide and 2 deep; in how many days 9 hours long, can 24 men dig a trench of 420 yards long, 5 wide and 3 deep?

The solid content of the first trench is $210 \times 3 \times 2 = 1260$ solid feet: and that of the second is $420 \times 5 \times 3 = 6300$ solid feet.

Now, 252 men in 55 hours dig 1260 solid feet:
whence, 1 man in 55 hours digs 5 solid feet:
and 24 men in 55 hours dig 120 solid feet:

therefore, 24 men in $55 \times 52\frac{1}{2}$ hours dig 6300 solid feet: and consequently, $\frac{55 \times 52\frac{1}{2}}{9} = 320\frac{5}{6}$ days of 9 hours each is the time required.

(5) If 10 men in 3 days reap a field the length of which is 1200 feet and the breadth 800 feet; what is the length of a field whose breadth is 1000 feet which 12 men can reap in four days?

Here, 10 men in 3 days reap 1200×800 square feet:
and 1 man in 3 days reaps 120×800 square feet:
also, 1 man in 1 day reaps 40×800 square feet:

whence, 12 men in 1 day reap 480×800 square feet:
and 12 men in 4 days reap 1920×800 square feet:
but $1920 \times 800 = 1536000 = 1536 \times 1000$ square feet:
whence, it follows that the required length is 1536 feet:

(6) If a pipe of 6 inches bore discharge a certain quantity of fluid in 4 hours: in what time will 4 pipes each of 3 inches bore, discharge twice that quantity?

If 1 denote the quantity of fluid discharged by the first pipe in 4 hours, we have $\frac{1}{4}$ = quantity discharged by it in 1 hour: but the quantities discharged by the pipes are as the *areas* of their sections and therefore as the *squares* of their diameters: whence,

$$\tfrac{1}{4} : \text{quantity discharged by one of the second set in 1 hour} :: 6^2 : 3^2 :: 4 : 1;$$

and therefore the quantity discharged by one of these pipes in 1 hour $= \frac{1}{16}$:

hence, the quantity by 4 such pipes in 1 hour $= \frac{1}{4}$; and therefore the quantity discharged by these 4 pipes in 8 hours $= 2$, or twice the quantity discharged by the first in 4 hours: that is, 8 hours is the time required.

(7) If a beam which is 10in. wide, 8in. deep and 5ft. 6in. long, weigh 8cwt. 1qr.: find the length of another beam of the same kind, the end of which is a square foot, which shall weigh 1 ton.

The volume of the first beam $= 10 \times 8 \times 66$ solid inches, and that of the second beam $= 12 \times 12 \times$ the required length $= 144x$ solid inches, suppose: also, the weights are proportional to the *volumes* or *masses*, and therefore we have

$$8\tfrac{1}{4}\text{cwt.} : 20\text{cwt} :: 10 \times 8 \times 66 : 144x:$$

$$\text{whence, } x = \frac{20 \times 10 \times 8 \times 66}{8\frac{1}{4} \times 144} = 88\tfrac{8}{9}\text{in.} = 7\text{ft. } 4\tfrac{8}{9}\text{in.}$$

(8) If a ball, whose diameter is 2 inches weigh 5lbs.; what must be the diameter of another ball of the same substance which shall weigh 78.125lbs.?

Since the weights are proportional to the *volumes*, and therefore to the *cubes* of the diameters, we have

5 lbs. : 78.125 lbs. :: 2^3 : (the required diameter)^3;

whence, (the required diameter)$^3 = \dfrac{8 \times 78.125}{5} = 125$:

and the required diameter $= \sqrt[3]{125} = 5$ inches.

(9) A rectangular court the sides of which are 300 feet and 200 feet has a walk 20 feet wide cut off from it on every side: compute the area of the walk and compare it with that of the court.

The area of the whole court $= 300 \times 200 = 60000$ square feet: also, since the dimensions are diminished 20 feet on *each* side by the walk, the area of the remaining portion

$= (300 - 40) \times (200 - 40) = 260 \times 160 = 41600$ square feet: whence, the area of the walk $= 60000 - 41600 = 18400$ square feet: and the walk therefore takes up

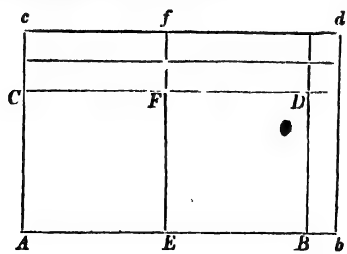
$$\frac{18400}{60000} = \frac{184}{600} = \frac{46}{150} = \frac{23}{75}\text{ths of the court.}$$

(10) Multiply 2 feet 1 inch by 1 foot 2 inches: and explain the meaning of the terms of the result by a *geometrical* construction.

Here by the rule of Article (212), we have

$$\begin{array}{rrr} 2^f \,.& 1' & \\ 1 \,.& 2 & \\ \hline 2 \,.& 1 & \\ & 4 \,.& 2 \\ \hline 2^f \,.& 5' \,.& 2'' \end{array}$$

To explain this result geometrically, if $AB = 2$ feet, $Bb = 1$ inch, $AC = 1$ foot, $Cc = 2$ inches and the construction be completed as below:



$$\text{we have } Ab = AB + Bb = 2^f \,.\, 1':$$
$$Ac = AC + Cc = 1^f \,.\, 2':$$
$$\text{also, } Ad = AD + Db + Dc + Dd:$$
$$\text{but } AD = AB \times AC = 2 \times 1 = 2 \text{ square feet}:$$
$$Db + Dc = 1^f \times 1' + 2^f \times 2' = 1' + 4' = 5 \text{ superficial primes}:$$
$$\text{and } Dd = Bb \times Cc = 1' \times 2' = 2 \text{ square seconds}:$$

that is, the entire product is $2^f \,.\, 5' \,.\, 2''$ superficial measure as before found: and the diagram shews clearly what is meant by each of the denominations of the result; namely, that a superficial prime is a *foot* in length and an *inch* in breadth; a superficial second is an *inch* in length and an *inch* in breadth, or a square inch: &c.

Hence, it appears that the product of *two* numbers of *feet* retains their common *denomination*, whilst in the *Duodecimal Subdivisions* the denomination of the product of two quantities of *different* names is ascertained by the *addition* of the numbers expressing the denominations of the quantities themselves: and this conclusion is sometimes embodied in the form of a *verbal* rule.

APPENDIX.

I. NOTATION AND NUMERATION.

1. It seems probable that the necessities of the human race would at a very early period suggest some method of *counting* or *reckoning*, as well as of *registering* the results of such processes: and the *instruments* employed, which in our language would be called *Counters*, might at any time convey to the mind a very distinct and clear idea of a number which did not consist of *many* individuals. Without entering into any historical account of the different Systems of Notation which have been used in different nations, or hazarding any conjectures as to the circumstances in which they may have had their origin, it is deemed sufficient for our present purpose to pass on immediately to the Notation now in use, which is fully explained in the first chapter of this work.

2. The characters 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, are said to have been transmitted to us from the *Arabians*, who again are supposed to have received them from the *Hindoos*, though in forms considerably different from those in which they are now written: and the word *Digit* usually applied to them, denoting a *Finger*, seems to point out the means originally employed in estimating numerical magnitudes, the number 10, which is called the *Base* or *Radix* of the system, and by which the *local values* of the *digits* are regulated, being that of the *Fingers* of *both* hands. The Notation appears to be as *complete* and *convenient* as can well be imagined, and in its present state may certainly be regarded as one of the greatest and most successful efforts of human ingenuity ever exhibited to the world.

The reader who is desirous of full information upon this subject should peruse *Professor Leslie's* interesting Work, entitled *The Philosophy of Arithmetic*.

3. In reference to what was said in Article (14), it may be proper to observe that the method of proceeding differs from that adopted by the *French* and some other *Foreign* Arithmeticians, who adhere throughout to divisions of *three* figures, according to the principle of Article (11), and after the division of *Millions*, proceed directly to that of *Billions*, tens of *Billions*, and hundreds of *Billions*: then to *Trillions*, tens of *Trillions*, and hundreds of *Trillions*, and so on: and this method certainly possesses some advantages in point of simplicity; but as numbers of these magnitudes are not of very frequent occurrence, it has not been thought necessary to depart from the *Notation* and *Nomenclature* established in this country.

In the following schemes it is seen how they differ.

English Nomenclature.

	Billions.	Millions.	Units.
&c.	987654	321987	654321

where each division consists of *six* figures: and it may be extended towards the left hand as far as we please.

French Nomenclature.

	Quadrillions.	Trillions.	Billions.	Millions.	Thousands.	Units.
&c.	987	654	321	987	654	321

each division consisting of *three* figures: and it is evident that as far as hundreds of millions are concerned, there is no difference whatever in the *reading* or *enumerating* of numbers in the two methods.

II. ADDITION AND SUBTRACTION.

4. The idea of number implies a capability of *increase* or *decrease*, the former of which is produced by the *operation* of *Addition*, and the latter by that of *Subtraction*: and a set of *Counters*, here represented by *units*,

will be of use in explaining the grounds upon which these operations are established.

Thus, suppose we wish to add *five* and *seven* together, then we have the following *parcels* of counters to represent them:

1, 1, 1, 1, 1, and 1, 1, 1, 1, 1, 1, 1;

and if these be *added* together, or collected into *one* parcel, their sum will evidently be represented by

1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,

which may again be put in the form

1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1;

and this shews the result to be *ten* together with *two*, or *twelve*: that is, the *sum* of *five* and *seven* is *twelve*.

Hence, in a system of counters there is in fact *no operation* to perform, as it is sufficient merely to *collect* or *combine* the counters into one group: and there would be no necessity for committing to memory the sum of two numbers as in our system, except so far as the *name* of that sum is concerned.

The same might manifestly be done with *more* and *larger* numbers, and it furnishes the *definition* of the operation of *Addition* given in Article (21).

5. To subtract *six* from *nine*, implying that of *nine* individuals, *six* are to be taken away, we must have at first *nine* counters, as

1, 1, 1, 1, 1, 1, 1, 1, 1,

which may be formed into the two *parcels*,

1, 1, 1, 1, 1, 1, and 1, 1, 1;

and if we withdraw *six* of these, or remove the *first* parcel, we have only *three* counters left, denoted by

1, 1, 1:

and thus we see that if *six* be *subtracted* from *nine*, the *remainder* will be *three*.

Here, the *withdrawal* of the *larger* parcel, or the *removal* of the *former* parcel, is the only process employed, and it needs no effort of the *mind* to perform it.

The operation of *Subtraction* is entirely founded upon this process, has always a *tacit* reference to it, and takes its *definition* from it, as is seen in Article (26).

6. The student will perceive that the *performances* of these two operations is not *facilitated* by the modern notation, except as to the *writing* and *reading* of the results. On the contrary, they are rendered considerably more *difficult*, and require *Rules* and *Directions* to work by, which have already been laid down in Articles (22) and (27): they however *depend* upon a system of counters, owe their *origin* entirely to it, and may at any time be effected by means of it.

Thus, using the arithmetical *signs* for the operations of Addition and Subtraction, we have

$$3 = 1 + 1 + 1:$$

$$5 = 1 + 1 + 1 + 1 + 1:$$

$$\text{whence, } 3 + 5 = (1 + 1 + 1) + (1 + 1 + 1 + 1 + 1)$$
$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$
$$= 8,$$

by omitting the brackets, which were introduced merely to keep the two parcels distinct from each other, and representing the *aggregate* or *assemblage* of units by its proper symbol: and it is here shewn that it is immaterial in what *order* the numbers to be added together are taken.

$$\text{Again, } 7 = 1 + 1 + 1 + 1 + 1 + 1 + 1:$$
$$4 = 1 + 1 + 1 + 1:$$

whence, if from the former of these be withdrawn or removed what is *equivalent* to the latter, there will remain $1 + 1 + 1$ or 3: and thus we have $7 - 4 = 3$.

7. Although in the operations of Addition and Subtraction as treated of in the text, it has been found convenient to commence at the *right* hand and proceed towards the *left*, the use of the arithmetical signs will enable us to perform the same operations in any order we may choose: thus, to find the sum and difference of 1345 and 274, we have

$$
\begin{aligned}
1345 &= 1000 + 300 + 40 + 5 \\
274 &= \qquad\quad 200 + 70 + 4 \\
\text{and the sum} &= 1000 + 500 + 110 + 9 \\
&= 1000 + 500 + 100 + 10 + 9 \\
&= 1000 + (500 + 100) + 10 + 9 \\
&= 1000 + 600 + 10 + 9 \\
&= 1619:
\end{aligned}
$$

$$
\begin{aligned}
\text{also, } 1345 &= 1000 + 300 + 40 + 5 \\
&= 1000 + 200 + 100 + 40 + 5 \\
&= 1000 + 200 + 140 + 5 \\
274 &= \qquad\quad 200 + 70 + 4 \\
\text{and the difference} &= 1000 + 0 + 70 + 1 \\
&= 1071.
\end{aligned}
$$

From these processes, which have a close resemblance to the method of reckoning by counters, we cannot but see, that by a *slight* exercise of the *mind* and the *memory*, much *real* labour is saved by means of the rules in the text, not to mention the *prolixity* of operation as well as the *number* of figures that would be required for larger magnitudes than those which have been used to establish this conclusion.

III. MULTIPLICATION AND DIVISION.

8. The operation intended by the word *Multiplication*, is defined in Article (31) of the text: and in the first place we will shew that the conclusions which it leads to, may be safely depended upon, as far as the *order* of the *factors* may influence the *product*.

To multiply 4 by 3, we have to repeat 4 or $1 + 1 + 1 + 1$, *three times*, and the product will be

$$
\begin{aligned}
&(1 + 1 + 1 + 1) + (1 + 1 + 1 + 1) + (1 + 1 + 1 + 1) \\
&= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
&= (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1),
\end{aligned}
$$

which is manifestly $1 + 1 + 1$ or 3, *four times* repeated: that is, *three* times *four* is the same as *four* times *three*.

By reasoning of this kind it is made to appear that the product has a *similar* or *symmetrical* relation to both its factors, because it remains the same when the multiplicand is made the multiplier, and the multiplier becomes the multiplicand.

9. If *each* of the *units* represent a *man*, or other *specified* individual, the process employed above is no longer admissible, as the result becomes unintelligible, or admits of no *numerical* interpretation.

That is, the multiplication of *concrete* magnitudes *as such*, is altogether impossible: but whenever it is applied to them, they are first supposed to be *divested* of their *concrete* character, or to be *abstracted:* and when the operation has been performed upon the corresponding *abstract* magnitudes, we have only to *explain* or *interpret* when possible, the meaning of the result.

Thus, if the factors were £7. and £8., we could easily multiply together the abstract numbers 7 and 8, whose product is 56; but the *denomination* of this *result* as the product of £7. and £8. cannot be ascertained, and the *operation* is altogether *absurd.*

To multiply £7. by the abstract number 8, the *correct* method of proceeding will be to consider the 7 *abstracted,* to find the product of the numbers 7 and 8: and the product 56 must then be interpreted in consistency with the nature of the question proposed: and it is seen immediately, that £7. being *repeated 8 times,* amounts to £56.: so that, whilst the multiplicand may be a *concrete* magnitude, the multiplier must be an *abstract* one, because the *number* of parcels of £7. each, is in no way dependent upon the *species* of the individuals contained in any one of them, but would remain the same were any other species or number of individuals to be repeated *eight* times.

The same observations will be applicable when the *species* of the concrete magnitudes are *different;* thus, if a person walk for 5 hours at the rate of 4 miles an hour, we find the product of the numbers 4 and 5 to be 20, which must, from the circumstances of the case, be 20 *miles* the whole distance travelled, and not 20 *hours,* because such an interpretation would necessarily

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